One-Bit Phase Retrieval

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Outline

Introduction

1-Bit Learning / Estimation

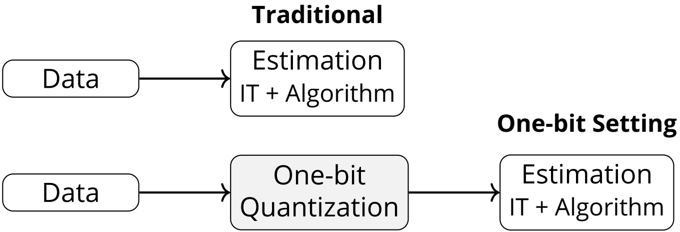
1-bit quantization

$$\mathcal{Q}:\mathbb{R}\to\{-1,1\}$$

is ubiquitous in signal processing, distributed learning, ...

Theoretical problems for applied mathematicians / statisticians ?

Learning / Estimation / Signal Recovery from 1-Bit Data !



Goal: understand the information-theoretic (IT) limits and devise efficient algorithms.

1-Bit Learning / Estimation

Many 1-bit estimation problems are well studied:

- 1-bit compressed sensing (Boufounos and Baraniuk, 2008)
  recovery of k-sparse $\mathbf{x}$ from $\mathbf{y} = sign(\mathbf{A}\mathbf{x})$
- 1-bit matrix completion (Davenport et al., 2014)
  estimate of low-rank $\mathbf{M}$ from its 1-bit entries
- 1-bit distributed mean estimation (Kipnis and Duchi, 2022)
  estimate $\mu = \mathbb{E}\mathbf{X}_i$ from $\mathcal{Q}_{1b}(\mathbf{X}_1), \ldots, \mathcal{Q}_{1b}(\mathbf{X}_n)$
- 2-bit covariance estimation (Dirksen et al., 2022)
  estimate $\mathbf{\Sigma} = \mathbb{E}(\mathbf{X}_i\mathbf{X}_i^T)$ from $\mathcal{Q}_{2b}(\mathbf{X}_1), \ldots, \mathcal{Q}_{2b}(\mathbf{X}_n)$
- Binary GLMs (Matsumoto and Mazumdar, 2025)
  logistic regression, probit regression
- Many others

1-Bit Phase Retrieval (1bPR)

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a known Gaussian matrix.

Phase Retrieval: (lose all phase information)

- recovery of $\mathbf{x} \in \mathbb{R}^n$ from

$$\mathbf{y} = |\mathbf{A}\mathbf{x}|$$

  or

$$y_i = |\mathbf{a}_i^T \mathbf{x}|, \quad i = 1, 2, ..., m$$

1-Bit phase retrieval:

- how to quantize $y_i = |\mathbf{a}_i^T \mathbf{x}|$ to ± 1? $\longrightarrow$ fix $\tau > 0$
- recovery of $\mathbf{x}$ from

$$\mathbf{y} = sign(|\mathbf{A}\mathbf{x}| - \tau)$$

  or

$$y_i = sign(|\mathbf{a}_i^T \mathbf{x}| - \tau), \quad i = 1, ..., m$$

- Suppose $\alpha \leq \|\mathbf{x}\|_2 \leq \beta$!
- lose all phase and then further lose nearly all magnitude !

1-Bit Compressed Sensing (1bCS)

1bCS: recovery of $\mathbf{x} \in \Sigma_k^{n,*} = \Sigma_k^n \cap S^{n-1}$ from $\mathbf{y} = sign(\mathbf{Ax})$

- **Information-theoretic problem:** how well can we recover $\mathbf{x}$?
- Jacques et al. (2013) shows

$$\overbrace{\underbrace{\inf_{\hat{\mathbf{x}}=\hat{\mathbf{x}}(\mathbf{A},\mathbf{y})} \sup_{\mathbf{x}\in\Sigma_k^{n,*}} \|\hat{\mathbf{x}}-\mathbf{x}\|_2 \asymp \frac{k}{m}}_{\Omega(\frac{k}{m})\text{: counting}}}^{\tilde{O}(\frac{k}{m})\text{: intractable program!}}, \quad \text{up to log factor}$$

- High-dimensional linear regression $\mathbf{y} = \mathbf{Ax} + N(0, \sigma^2 I_m)$:

$$\inf_{\hat{\mathbf{x}}=\hat{\mathbf{x}}(\mathbf{A},\mathbf{y})} \sup_{\mathbf{x}\in\Sigma_k^{n,*}} \|\hat{\mathbf{x}}-\mathbf{x}\|_2 \asymp \sigma\sqrt{\frac{k\log(en/k)}{m}}$$

- **Algorithmic problem:** how to use an efficient algorithm to achieve the optimal rate $\tilde{O}(\frac{k}{m})$?
- Recently resolved by Matsumoto and Mazumdar (2024) who showed that NBIHT attains $\|\hat{\mathbf{x}}_{nbiht} - \mathbf{x}\|_2 = \tilde{O}(\frac{k}{m})$

Algorithmic Problem

1bPR: recovery of $\alpha \le \|\mathbf{x}\|_2 \le \beta$ from $\mathbf{y} = sign(|\mathbf{A}\mathbf{x}| - \tau)$

- Information-theoretic optimal rate: (easy to resolve)

$$\overbrace{\underbrace{\inf_{\hat{\mathbf{x}}=\hat{\mathbf{x}}(\mathbf{A},\mathbf{y},\tau)} \sup_{\mathbf{x}\in\mathbb{A}_\alpha^\beta} \min\{\|\hat{\mathbf{x}}-\mathbf{x}\|_2, \|\hat{\mathbf{x}}+\mathbf{x}\|_2\}}_{\Omega(\frac{n}{m}):\text{ counting}}}^{\tilde{O}(\frac{n}{m}):\text{ intractable program!}} \asymp \frac{n}{m}, \quad \text{up to log factor} \tag{1.1}$$

where $\min\{\|\hat{\mathbf{x}}-\mathbf{x}\|_2, \|\hat{\mathbf{x}}+\mathbf{x}\|_2\} := dist(\hat{\mathbf{x}},\mathbf{x})$

- **Algorithmic problem:** (Our Focus!)

How to devise an efficient algorithm $\hat{\mathbf{x}}$ to attain $dist(\hat{\mathbf{x}},\mathbf{x}) = \tilde{O}(\frac{n}{m})$?

Optimal Algorithm

Phase retrieval via amplitude-flow

PR: recover of x from $\{y_i = |\mathbf{a}_i^T\mathbf{x}|\}_{i=1}^m$ (Zhang et al., 2017)

- Spectral initialization:
  - Compute the leading eigenvector $\hat{\mathbf{v}}_0$ of $\hat{\mathbf{S}} = \frac{1}{m}\sum_{i=1}^m y_i\mathbf{a}_i\mathbf{a}_i^T$
  - Let

$$\mathbf{x}_0 = \underbrace{\sqrt{\frac{\pi}{2}}\Big(\frac{1}{m}\sum_{i=1}^m y_i\Big)}_{\text{norm estimate}} \cdot \overbrace{\hat{\mathbf{v}}_0}^{\text{direction estimate}}$$

- Gradient descent:
  - Consider amplitude-based loss

$$\mathcal{L}(\mathbf{u}) = \frac{1}{2m}\sum_{i=1}^m \big(|\mathbf{a}_i^T\mathbf{u}| - y_i\big)^2$$

  and perform

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \eta\partial\mathcal{L}(\mathbf{u}) = \mathbf{x}_t - \frac{\eta}{m}\sum_{i=1}^m \big(|\mathbf{a}_i^T\mathbf{u}| - y_i\big)\, sign(\mathbf{a}_i^T\mathbf{u})\mathbf{a}_i$$

Spectral Initialization (Standard)

This is very analogous to phase retrieval:

- (Direction estimation) Compute the leading eigenvector $\hat{\mathbf{v}}_0$ of

$$\hat{\mathbf{S}} = \frac{1}{m} \sum_{i=1}^{m} y_i \mathbf{a}_i \mathbf{a}_i^T$$

- (Norm estimation) We look at $\hat{\lambda}_{\mathbf{x}} = \frac{1}{m} \sum_{i=1}^{m} \mathbb{1}(y_i = 1)$ and let

$$\hat{\lambda}_0 = \frac{-\tau}{\Phi^{-1}(\hat{\lambda}_{\mathbf{x}}/2)}, \quad \text{where } \Phi \text{ is the C.D.F. of } N(0,1)$$

- We use

$$\hat{\mathbf{x}}_0 = \hat{\lambda}_0 \hat{\mathbf{v}}_0$$

- Spectral initialization is compatible with coarsely quantized measurements

Gradient Descent

Motivated by NBIHT, we use ReLU loss:

- Hamming distance loss

$$\mathcal{L}_{hd}(\mathbf{u}) = \frac{1}{m}\sum_{i=1}^{m} \mathbb{1}\big(sign(|\mathbf{a}_i^T\mathbf{u}| - \tau) \neq y_i\big) = \frac{1}{m}\sum_{i=1}^{m} \mathbb{1}\big(-y_i(|\mathbf{a}_i^T\mathbf{u}| - \tau) \geq 0\big)$$

- ReLU loss: $\mathbb{1}(a \geq 0) \to \mathrm{Relu}(a) := \max\{a, 0\}$

$$\mathcal{L}_{relu}(\mathbf{u}) = \frac{1}{m}\sum_{i=1}^{m} \mathrm{Relu}\big(-y_i(|\mathbf{a}_i^T\mathbf{u}| - \tau)\big)$$

- **Why Relu relaxation?** (remains nonconvex, nonsmooth !)
  - weak tightness:

$$\mathcal{L}_{hd}(\hat{\mathbf{x}}) = 0 \iff \mathcal{L}_{relu}(\hat{\mathbf{x}}) = 0$$

  - subgradient:

$$\text{discrete} \to \text{continuous} \to \text{subgradient}$$

Gradient Descent

- By $\mathrm{Relu}(a) = \frac{1}{2}(a + |a|)$ and $y_i = sign(|\mathbf{a}_i^T \mathbf{x}| - \tau)$,

$$\mathcal{L}_{relu}(\mathbf{u}) = \frac{1}{2m} \sum_{i=1}^{m} \Big[||\mathbf{a}_i^T \mathbf{u}| - \tau| - y_i(|\mathbf{a}_i^T \mathbf{u}| - \tau) \Big]$$

  and the subgradient is

$$\partial \mathcal{L}_{relu}(\mathbf{u}) = \underbrace{\frac{1}{2m} \sum_{i=1}^{m} \big(sign(|\mathbf{a}_i^T \mathbf{u}| - \tau) - sign(|\mathbf{a}_i^T \mathbf{x}| - \tau) \big) sign(\mathbf{a}_i^T \mathbf{u}) \mathbf{a}_i}_{:= \mathbf{h}(\mathbf{u}, \mathbf{x}) \quad \text{subgradient at } \mathbf{u} \text{ when } \mathbf{x} \text{ is desired}}$$

- Gradient descent

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \eta \cdot \mathbf{h}(\mathbf{x}_t, \mathbf{x}),$$

  while the following question remains:

How to choose step size η ?

Optimal Convergence via AIC

Guarantee for Spectral Initialization

Theorem (Initialization guarantee)

If $m \gtrsim n$, then w.h.p.,

$$dist(\mathbf{x}_0, \mathbf{x}) \lesssim \sqrt{\frac{n}{m}}$$

- **Proof ingredients (standard)**
  - Davis-Kahan $\sin \Theta$ Theorem
  - Operator norm concentration
- **Takeaway:** for any small $\delta_0 > 0$,

We can focus on $\{\mathbf{u} : dist(\mathbf{u}, \mathbf{x}) \leq \delta_0\}$ as long as $m \gtrsim \delta_0^{-2} n$!

Approximate Invertibility Condition (AIC)

Our **structured condition** to convergence:

Definition (Local AIC)

Fix $\mathbf{A}$ and τ and given $\beta_1 \geq \alpha_1 > 0$ and $\delta_1, \delta_2, \delta_3, \delta_4 \geq 0$, we say

$$\underbrace{\mathbf{h}(\mathbf{u},\mathbf{v}) := \frac{1}{2m}\sum_{i=1}^{m}\big(sign(|\mathbf{a}_i^T\mathbf{u}|-\tau) - sign(|\mathbf{a}_i^T\mathbf{v}|-\tau)\big)\,sign(\mathbf{a}_i^T\mathbf{u})\mathbf{a}_i}_{\text{gradient at } \mathbf{u} \text{ when } \mathbf{v} \text{ is desired}}$$

satisfies Local AIC($\alpha_1, \beta_1, \delta_1, \delta_2, \delta_3, \delta_4$) with step size η, if

$$\|\overbrace{\mathbf{u}-\mathbf{v}}^{\text{ideal step}} - \overbrace{\eta\cdot\mathbf{h}(\mathbf{u},\mathbf{v})}^{\text{actual step}}\|_2 \leq \overbrace{\delta_1\|\mathbf{u}-\mathbf{v}\|_2 + \sqrt{\delta_2\|\mathbf{u}-\mathbf{v}\|_2} + \delta_3}^{\text{approximation error terms}}$$

$$\forall\, \mathbf{u},\mathbf{v} \in \mathbb{A}_{\alpha_1}^{\beta_1} := \{\mathbf{u} : \alpha_1 \leq \|\mathbf{u}\|_2 \leq \beta_1\} \text{ obeying } \underbrace{\|\mathbf{u}-\mathbf{v}\|_2 \leq \delta_4}_{\text{localization !}}$$

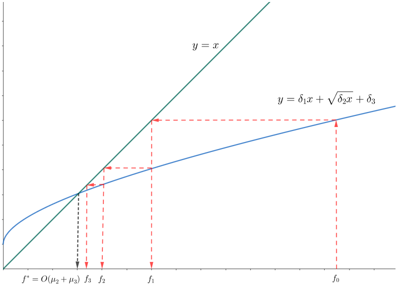
Why Local AIC is Useful?

Let $\mathbf{u} = \mathbf{x}_t$, $\mathbf{v} = \mathbf{x}$, $\mathbf{x}_{t+1} = \mathbf{x}_t - \eta \cdot \mathbf{h}(\mathbf{x}_t, \mathbf{x})$,

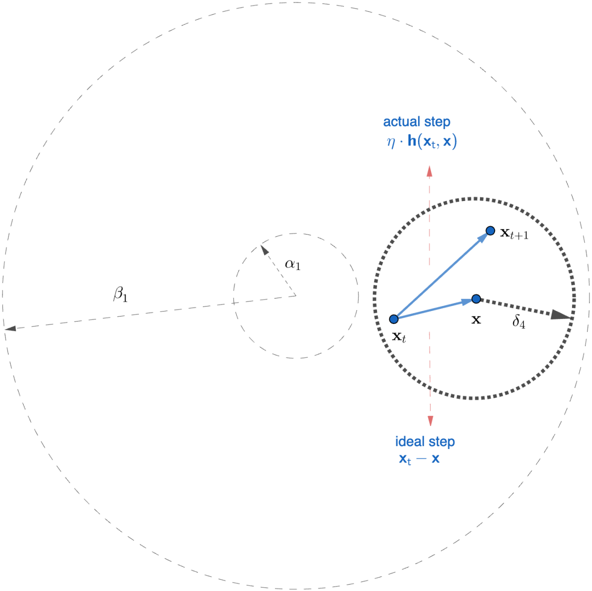
$$\| \underbrace{\overbrace{\mathbf{x}_t - \mathbf{x}}^{\text{ideal step}} - \overbrace{\eta \cdot \mathbf{h}(\mathbf{x}_t, \mathbf{x})}^{\text{actual step}}}_{=\mathbf{x}_{t+1} - \mathbf{x}} \|_2 \le \delta_1 \|\mathbf{x}_t - \mathbf{x}\|_2 + \sqrt{\delta_2 \|\mathbf{x}_t - \mathbf{x}\|_2} + \delta_3$$

$$\Longrightarrow f_{t+1} \le \delta_1 f_t + \sqrt{\delta_2 f_t} + \delta_3 \quad \text{where} \quad f_t = \|\mathbf{x}_t - \mathbf{x}\|_2$$

$$\Longrightarrow \{f_t\}_{t \ge 0} \searrow O(\delta_2 + \delta_3) \text{ as long as } \delta_1 < 1$$



Graphical Illustration



The Desired Local AIC

Claim. To show that our algorithm $\hat{\mathbf{x}}$ achieves

$$\|\hat{\mathbf{x}} - \mathbf{x}\|_2 = \tilde{O}\left(\frac{n}{m}\right)$$

under $m = \tilde{\Omega}(n)$, we only need to prove Local AIC with

- $\alpha_1 = \alpha$, $\beta_1 = \beta$
- $\delta_1 < 1$
  (contraction: η should be chosen to render this)
- $\delta_2, \delta_3 = \tilde{O}(\frac{n}{m})$
  (this dictates the near-optimal error rate; cf. (1.1))
- $\delta_4 = \tilde{\Theta}(1)$
  (this dictates the near-optimal sample complexity – recall that we need

$$m \gtrsim \delta_4^{-2} n \quad \text{(very small } \delta_4 \text{ can worsen the sample complexity)}$$

to ensure $\|\mathbf{x}_0 - \mathbf{x}\|_2 \leq \delta_4$)

The Desired Local AIC

It remains to prove, under

$$\underbrace{\alpha = 1/2,\ \beta = 2,\ \tau = 1}_{\text{just for concreteness}}$$

that

$$\Big\|\mathbf{u}-\mathbf{v}-\overbrace{\eta\cdot\mathbf{h}(\mathbf{u},\mathbf{v})}^{\eta\ \text{to be determined}}\Big\|_2 \le \overbrace{\delta_1}^{\delta_1<1}\|\mathbf{u}-\mathbf{v}\|_2+\sqrt{\tilde{O}(\frac{n}{m})\|\mathbf{u}-\mathbf{v}\|_2}+\tilde{O}(\frac{n}{m})$$

$$\underbrace{\forall \mathbf{u},\mathbf{v}\in\mathbb{A}^2_{1/2}\ \text{ obeying }\ \|\mathbf{u}-\mathbf{v}\|_2\le\delta_4=\tilde{\Theta}(1)}_{\text{covering to get uniformity}}$$

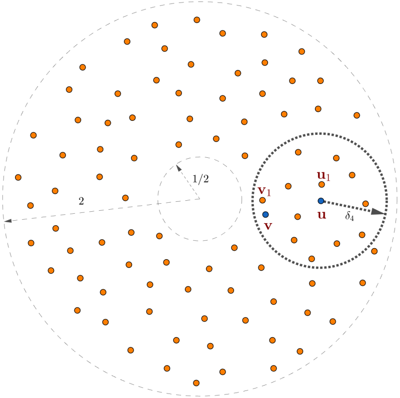
$$\mathbf{h}(\mathbf{u},\mathbf{v})=\frac{1}{2m}\sum_{i=1}^{m}\big(sign(|\mathbf{a}_i^T\mathbf{u}|-1)-sign(|\mathbf{a}_i^T\mathbf{v}|-1)\big)sign(\mathbf{a}_i^T\mathbf{u})\mathbf{a}_i$$

Prove AIC via Covering Argument

Proof Architecture: Covering Argument

- $\mathcal{N}_r$: minimal r-net of $\mathbb{A}^2_{1/2}$ of size bounded by $|\mathcal{N}_r| \leq (\frac{6}{r})^n$
- For any $\mathbf{u}, \mathbf{v} \in \mathbb{A}^2_{1/2}$, $\|\mathbf{u} - \mathbf{v}\|_2 \leq \delta_4$, let

$$\mathbf{u}_1 := \arg\min_{\mathbf{w} \in \mathcal{N}_r} \|\mathbf{w} - \mathbf{u}\|_2 , \quad \mathbf{v}_1 := \arg\min_{\mathbf{w} \in \mathcal{N}_r} \|\mathbf{w} - \mathbf{v}\|_2$$



Large-distance regime: $\|\mathbf{u}_1 - \mathbf{v}_1\|_2 \geq r$

Finally, we will use

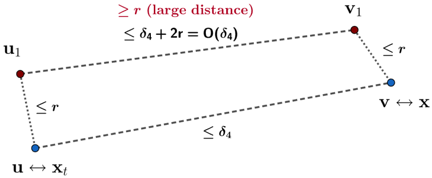
$$r = \tilde{O}(\frac{n}{m}) \ll \delta_4 = \tilde{O}(1)$$

We call

$$\|\mathbf{u}_1 - \mathbf{v}_1\|_2 \geq r$$

the large-distance regime, which occurs for instance when

$$r \ll \|\mathbf{u} - \mathbf{v}\|_2 \leq \delta_4$$



Optimization perspective: far from optimal & need contraction

What to Bound under Large Distance?

Passing to the net,

$$\|\mathbf{u}-\mathbf{v}-\eta\mathbf{h}(\mathbf{u},\mathbf{v})\|_2$$
$$\leq \|\mathbf{u}-\mathbf{u}_1\|_2+\|\mathbf{v}-\mathbf{v}_1\|_2+\|\mathbf{u}_1-\mathbf{v}_1-\eta\mathbf{h}(\mathbf{u}_1,\mathbf{v}_1)\|_2+\eta\|\mathbf{h}(\mathbf{u},\mathbf{v})-\mathbf{h}(\mathbf{u}_1,\mathbf{v}_1)\|_2$$
$$\leq 2r+\overbrace{\underbrace{\|\mathbf{u}_1-\mathbf{v}_1-\eta\cdot\mathbf{h}(\mathbf{u}_1,\mathbf{v}_1)\|_2}_{\substack{(\mathbf{u}_1,\mathbf{v}_1)\in\mathcal{N}^{(2)}_{r,\delta_4}:=\\ \{(\mathbf{p},\mathbf{q})\in\mathcal{N}_r\times\mathcal{N}_r:r\leq\|\mathbf{p}-\mathbf{q}\|_2\leq 2\delta_4\}}}^{\text{contraction term}}+\overbrace{\eta\cdot\|\mathbf{h}(\mathbf{u},\mathbf{v})-\mathbf{h}(\mathbf{u}_1,\mathbf{v}_1)\|_2}^{\text{approximation term}}$$

We only need to

1. Bound $\|\mathbf{p}-\mathbf{q}-\eta\cdot\mathbf{h}(\mathbf{p},\mathbf{q})\|_2$ for all $(\mathbf{p},\mathbf{q})\in\mathcal{N}^{(2)}_{r,\delta_4}$
   - Blessing: $|\mathcal{N}^{(2)}_{r,\delta_4}|\leq|\mathcal{N}_r|^2\leq(\frac{6}{r})^{2n}$ — bound for fixed $(\mathbf{p},\mathbf{q})$ + union bound
   - Hurdle: We need to choose η and understand how 'actual step' $\eta\mathbf{h}(\mathbf{p},\mathbf{q})$ approximates 'ideal step' $\mathbf{p}-\mathbf{q}$
2. Bound $\|\mathbf{h}(\mathbf{u},\mathbf{v})-\mathbf{h}(\mathbf{u}_1,\mathbf{v}_1)\|_2$ for all $\|\mathbf{u}-\mathbf{v}\|_2\leq\delta_4$
   - Hurdle: infinitely many $(\mathbf{u},\mathbf{v})$ — how to get uniformity ?

Small-distance regime: $\|\mathbf{u}_1 - \mathbf{v}_1\|_2 < r$

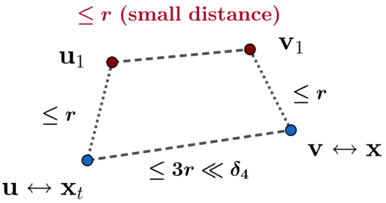
We call

$$\|\mathbf{u}_1 - \mathbf{v}_1\|_2 \le r$$

the small-distance regime, which implies

$$\|\mathbf{u} - \mathbf{v}\|_2 \le 2r + \|\mathbf{u}_1 - \mathbf{v}_1\|_2 \le 3r \ll \delta_4$$

Optimization perspective:
already optimal & need to stay in the optimal neighborhood



This regime is easy to analyze by parts of techniques in large-distance regime.

Contraction Term

Contraction Term

Goal: Choose η and bound

$$\|\mathbf{p}-\mathbf{q}-\eta\cdot\mathbf{h}(\mathbf{p},\mathbf{q})\|_2, \quad \forall\,(\mathbf{p},\mathbf{q}) \in \mathcal{N}^{(2)}_{r,\delta_4} = \{(\mathbf{p},\mathbf{q}) \in \mathcal{N}_r \times \mathcal{N}_r : r \le \|\mathbf{p}-\mathbf{q}\|_2 \le 2\delta_4\}$$

where

$$\mathbf{h}(\mathbf{p},\mathbf{q}) = \frac{1}{2m}\sum_{i=1}^{m} \underbrace{\left[sign(|\mathbf{a}_i^T\mathbf{p}|-1) - sign(|\mathbf{a}_i^T\mathbf{q}|-1)\right] sign(\mathbf{a}_i^T\mathbf{p})\,\mathbf{a}_i}_{:=X_i^{\mathbf{p},\mathbf{q}}}$$

A first index set $\mathbf{R}_{\mathbf{p},\mathbf{q}}$: m contributors, but only the ones in

$$\mathbf{R}_{\mathbf{p},\mathbf{q}} := \{i \in [m] : sign(|\mathbf{a}_i^T\mathbf{p}|-1) \ne sign(|\mathbf{a}_i^T\mathbf{q}|-1)\}$$

are nonzero

Simplify the Gradient

Reduction from $\mathbf{R}_{\mathbf{p},\mathbf{q}}$:

$$\mathbf{h}(\mathbf{p},\mathbf{q}) = \frac{1}{2m}\sum_{i\in\mathbf{R}_{\mathbf{p},\mathbf{q}}} X_i^{\mathbf{p},\mathbf{q}}\mathbf{a}_i$$

if $i \in \mathbf{R}_{\mathbf{p},\mathbf{q}}$,

$$\begin{aligned} X_i^{\mathbf{p},\mathbf{q}} &= [sign(|\mathbf{a}_i^T\mathbf{p}| - 1) - sign(|\mathbf{a}_i^T\mathbf{q}| - 1)]sign(\mathbf{a}_i^T\mathbf{p}) \\ &= 2sign(|\mathbf{a}_i^T\mathbf{p}| - |\mathbf{a}_i^T\mathbf{q}|)sign(\mathbf{a}_i^T\mathbf{p}) \\ &= 2sign(\mathbf{a}_i^T\mathbf{p} - \underbrace{[sign(\mathbf{a}_i^T\mathbf{p})sign(\mathbf{a}_i^T\mathbf{q})]}_{\text{further simplification?}}\mathbf{a}_i^T\mathbf{q}) \end{aligned}$$

[by $sign(a)sign(b) = sign(ab)$]

A second index set $\mathbf{L}_{\mathbf{p},\mathbf{q}}$: Define

$$\mathbf{L}_{\mathbf{p},\mathbf{q}} = \{i \in [m] : sign(\mathbf{a}_i^T\mathbf{p}) \neq sign(\mathbf{a}_i^T\mathbf{q})\}$$

So

$$sign(\mathbf{a}_i^T\mathbf{p})sign(\mathbf{a}_i^T\mathbf{q}) = \begin{cases} 1, & if \;\; i \notin \mathbf{L}_{\mathbf{p},\mathbf{q}} \\ -1, & if \;\; i \in \mathbf{L}_{\mathbf{p},\mathbf{q}} \end{cases}$$

Simplify the Gradient

Reductions from $\mathbf{R}_{\mathbf{p},\mathbf{q}}$, $\mathbf{L}_{\mathbf{p},\mathbf{q}}$:

$$X_i^{\mathbf{p},\mathbf{q}} = \begin{cases} 0, & \textit{if} \ \ i \notin \mathbf{R}_{\mathbf{p},\mathbf{q}} \\ 2\,\mathit{sign}(\mathbf{a}_i^T(\mathbf{p}-\mathbf{q})), & \textit{if} \ \ i \in \mathbf{R}_{\mathbf{p},\mathbf{q}} \setminus \mathbf{L}_{\mathbf{p},\mathbf{q}} \\ 2\,\mathit{sign}(\mathbf{a}_i^T(\mathbf{p}+\mathbf{q})), & \textit{if} \ \ i \in \mathbf{R}_{\mathbf{p},\mathbf{q}} \cap \mathbf{L}_{\mathbf{p},\mathbf{q}} \end{cases}$$

So $\mathbf{h}(\mathbf{p},\mathbf{q}) = \frac{1}{2m}\sum_{i\in\mathbf{R}_{\mathbf{p},\mathbf{q}}} X_i^{\mathbf{p},\mathbf{q}}\mathbf{a}_i$ simplifies to

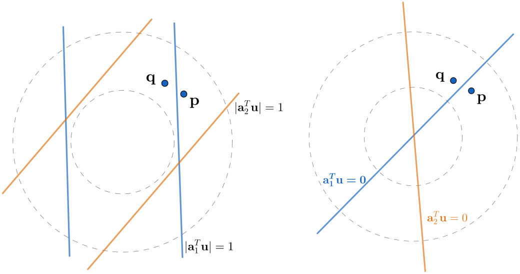
$$\mathbf{h}(\mathbf{p},\mathbf{q}) = \frac{1}{2m}\left(\sum_{i\in\mathbf{R}_{\mathbf{p},\mathbf{q}}\setminus\mathbf{L}_{\mathbf{p},\mathbf{q}}} 2\,\mathit{sign}(\mathbf{a}_i^T(\mathbf{p}-\mathbf{q}))\mathbf{a}_i + \sum_{i\in\mathbf{R}_{\mathbf{p},\mathbf{q}}\cap\mathbf{L}_{\mathbf{p},\mathbf{q}}} 2\,\mathit{sign}(\mathbf{a}_i^T(\mathbf{p}+\mathbf{q}))\mathbf{a}_i\right)$$

$$= \frac{1}{m}\left(\sum_{i\in\mathbf{R}_{\mathbf{p},\mathbf{q}}} \mathit{sign}(\mathbf{a}_i^T(\mathbf{p}-\mathbf{q}))\mathbf{a}_i - \sum_{i\in\mathbf{R}_{\mathbf{p},\mathbf{q}}\cap\mathbf{L}_{\mathbf{p},\mathbf{q}}} \mathit{sign}(\mathbf{a}_i^T(\mathbf{p}-\mathbf{q}))\mathbf{a}_i + \sum_{i\in\mathbf{R}_{\mathbf{p},\mathbf{q}}\cap\mathbf{L}_{\mathbf{p},\mathbf{q}}} \mathit{sign}(\mathbf{a}_i^T(\mathbf{p}+\mathbf{q}))\mathbf{a}_i\right)$$

$$= \underbrace{\frac{1}{m}\sum_{i\in\mathbf{R}_{\mathbf{p},\mathbf{q}}} \mathit{sign}(\mathbf{a}_i^T(\mathbf{p}-\mathbf{q}))\mathbf{a}_i}_{\substack{:=\mathbf{h}_1(\mathbf{p},\mathbf{q}) \\ \textbf{(main term – our focus)}}} + \underbrace{\frac{1}{m}\sum_{i\in\mathbf{R}_{\mathbf{p},\mathbf{q}}\cap\mathbf{L}_{\mathbf{p},\mathbf{q}}} \left[\mathit{sign}(\mathbf{a}_i^T(\mathbf{p}+\mathbf{q})) - \mathit{sign}(\mathbf{a}_i^T(\mathbf{p}-\mathbf{q}))\right]\mathbf{a}_i}_{\substack{:=\mathbf{h}_2(\mathbf{p},\mathbf{q}) \\ \textbf{(higher-order term)}}}.$$

Hyperplane tessellation

Geometry of $\mathbf{R}_{\mathbf{p},\mathbf{q}} = \{i : \mathit{sign}(|\mathbf{a}_i^T\mathbf{p}| - 1) \neq \mathit{sign}(|\mathbf{a}_i^T\mathbf{q}| - 1)\}$ and $\mathbf{L}_{\mathbf{p},\mathbf{q}} = \{i : \mathit{sign}(\mathbf{a}_i^T\mathbf{p}) \neq \mathit{sign}(\mathbf{a}_i^T\mathbf{q})\}$



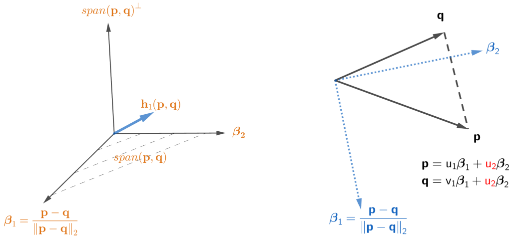
(a) $1 \in \mathbf{R}_{\mathbf{p},\mathbf{q}}, 2 \notin \mathbf{R}_{\mathbf{p},\mathbf{q}}$ (b) $1 \in \mathbf{L}_{\mathbf{p},\mathbf{q}}, 2 \notin \mathbf{L}_{\mathbf{p},\mathbf{q}}$

Why $\mathbf{h}_1(\mathbf{p},\mathbf{q})$ is main term and $\mathbf{h}_2(\mathbf{p},\mathbf{q})$ is negligible? $|\mathbf{R}_{\mathbf{p},\mathbf{q}} \cap \mathbf{L}_{\mathbf{p},\mathbf{q}}| \ll |\mathbf{R}_{\mathbf{p},\mathbf{q}}|$

$$\|\mathbf{p} - \mathbf{q} - \eta \cdot \mathbf{h}(\mathbf{p},\mathbf{q})\|_2 \leq \underbrace{\|\mathbf{p} - \mathbf{q} - \eta \cdot \mathbf{h}_1(\mathbf{p},\mathbf{q})\|_2}_{\text{our focus}} + \underbrace{\eta\|\mathbf{h}_2(\mathbf{p},\mathbf{q})\|_2}_{\text{negligible}}$$

Orthogonal Decomposition

Main Term: bound $\|\mathbf{p}-\mathbf{q}-\eta\cdot\mathbf{h}_1(\mathbf{p},\mathbf{q})\|_2$ for $(\mathbf{p},\mathbf{q})\in\mathcal{N}^{(2)}_{r,\delta_4}$
Orthogonal Decomposition: along 3 directions



so that

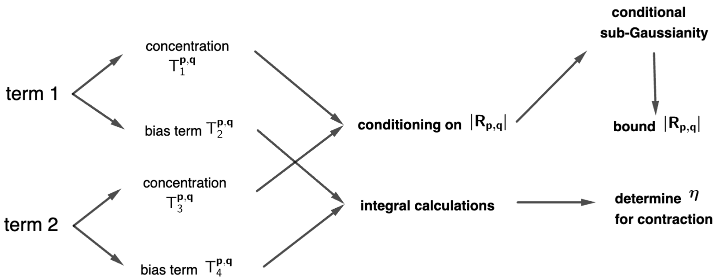
$$\mathbf{h}_1(\mathbf{p},\mathbf{q})=\langle\mathbf{h}_1(\mathbf{p},\mathbf{q}),\beta_1\rangle\beta_1+\langle\mathbf{h}_1(\mathbf{p},\mathbf{q}),\beta_2\rangle\beta_2 + \underbrace{\mathbf{h}_1(\mathbf{p},\mathbf{q})-\langle\mathbf{h}_1(\mathbf{p},\mathbf{q}),\beta_1\rangle\beta_1-\langle\mathbf{h}_1(\mathbf{p},\mathbf{q}),\beta_2\rangle\beta_2}_{:=\mathbf{h}_1^{\perp}(\mathbf{p},\mathbf{q})}$$

Orthogonal Decomposition

then

$$\|\mathbf{p}-\mathbf{q}-\eta\cdot\mathbf{h}_1(\mathbf{p},\mathbf{q})\|_2 \le \overbrace{\Big|\|\mathbf{p}-\mathbf{q}\|_2-\eta\langle\mathbf{h}_1(\mathbf{p},\mathbf{q}),\beta_1\rangle\Big|}^{\text{term 1}} + \underbrace{\eta|\langle\mathbf{h}_1(\mathbf{p},\mathbf{q}),\beta_2\rangle|}_{\substack{\text{term 2}\\ \text{similar to term 1}}} + \underbrace{\eta\|\mathbf{h}_1^{\perp}(\mathbf{p},\mathbf{q})\|_2}_{\substack{\text{term 3}\\ \text{centered \& similar}}}$$

Overview:



Example: Term 1

by centering $\big|\|\mathbf{p}-\mathbf{q}\|_2 - \eta\langle \mathbf{h}_1(\mathbf{p},\mathbf{q}), \beta_1\rangle\big|$,

$$\begin{aligned}
&\big|\|\mathbf{p}-\mathbf{q}\|_2 - \eta\langle \mathbf{h}_1(\mathbf{p},\mathbf{q}), \beta_1\rangle\big| \\
&= \Big|\|\mathbf{p}-\mathbf{q}\|_2 - \eta \cdot \frac{1}{m}\sum_{i\in\mathbf{R}_{\mathbf{p},\mathbf{q}}} |\mathbf{a}_i^T\beta_1|\Big| \\
&\le \overbrace{\eta\Big|\frac{1}{m}\sum_{i=1}^m \Big(|\mathbf{a}_i^T\beta_1|\mathbb{1}(i\in\mathbf{R}_{\mathbf{p},\mathbf{q}}) - \mathbb{E}\Big[|\mathbf{a}_i^T\beta_1|\mathbb{1}(i\in\mathbf{R}_{\mathbf{p},\mathbf{q}})\Big]\Big)\Big|}^{\text{concentration term } T_1^{\mathbf{p},\mathbf{q}}} \\
&+ \underbrace{\Big|\eta\cdot\mathbb{E}\big[|\mathbf{a}_i^T\beta_1|\mathbb{1}(i\in\mathbf{R}_{\mathbf{p},\mathbf{q}})\big] - \|\mathbf{p}-\mathbf{q}\|_2\Big|}_{\text{bias term } T_2^{\mathbf{p},\mathbf{q}}}
\end{aligned}$$

Concentration Term $T_1^{\mathbf{p},\mathbf{q}}$

Step 1 — Conditional Concentration $T_1^{\mathbf{p},\mathbf{q}}|\{|\mathbf{R}_{\mathbf{p},\mathbf{q}}| = r_{\mathbf{p},\mathbf{q}}\}$

1. Observe

$$T_1^{\mathbf{p},\mathbf{q}}|\{|\mathbf{R}_{\mathbf{p},\mathbf{q}}| = r_{\mathbf{p},\mathbf{q}}\} \overset{d}{=} \eta \underbrace{\left|\frac{1}{m}\sum_{i=1}^{r_{\mathbf{p},\mathbf{q}}}\left(Z_i^{\mathbf{p},\mathbf{q}} - \mathbb{E}[Z_i^{\mathbf{p},\mathbf{q}}]\right)\right|}_{Z_i^{\mathbf{p},\mathbf{q}} \overset{iid}{\sim} |\mathbf{a}_i^T\beta_1| \,\big|\, \{i \in \mathbf{R}_{\mathbf{p},\mathbf{q}}\}}$$

2. Show

$$\|Z_i^{\mathbf{p},\mathbf{q}}\|_{\psi_2} = O(1)$$

   1. Write the density function $f_{Z_i^{\mathbf{p},\mathbf{q}}}(z)$ of $Z_i^{\mathbf{p},\mathbf{q}}$
   2. Check the tails of $f_{Z_i^{\mathbf{p},\mathbf{q}}}(z)$

3. Reach

$$\mathbb{P}\left(T_1^{\mathbf{p},\mathbf{q}} \geq \frac{\eta\sqrt{r_{\mathbf{p},\mathbf{q}}}\,t}{m} \,\Big|\, |\mathbf{R}_{\mathbf{p},\mathbf{q}}| = r_{\mathbf{p},\mathbf{q}}\right) \leq 2\exp(-ct^2), \quad \forall t > 0$$

$$\Longrightarrow \mathbb{P}\left(T_1^{\mathbf{p},\mathbf{q}} \lesssim \frac{\eta\sqrt{r_{\mathbf{p},\mathbf{q}}\, n \log(r^{-1})}}{m} \,\Big|\, |\mathbf{R}_{\mathbf{p},\mathbf{q}}| = r_{\mathbf{p},\mathbf{q}}\right) \geq 1 - \underbrace{\exp\left(-4n\log\left(\frac{6}{r}\right)\right)}_{\text{to tolerate union bound over } \mathcal{N}^{(2)}_{r,\delta_4}}$$

Concentration Term $T_1^{\mathbf{p},\mathbf{q}}$

Step 2 — Get Rid of the Conditioning

1. Notice
$$|\mathbf{R}_{\mathbf{p},\mathbf{q}}| \sim \mathrm{Bin}(m, P_{\mathbf{p},\mathbf{q}})$$
where
$$P_{\mathbf{p},\mathbf{q}} := \mathbb{P}(i \in \mathbf{R}_{\mathbf{p},\mathbf{q}}) \asymp dist(\mathbf{p},\mathbf{q}) = \|\mathbf{p}-\mathbf{q}\|_2$$
2. By Chernoff bound,
$$|\mathbf{R}_{\mathbf{p},\mathbf{q}}| \lesssim m\|\mathbf{p}-\mathbf{q}\|_2, \quad w.p. \geq 1-\exp(-4n\log(\frac{6}{r}))$$

Step 3 — Concluding & Union Bound

We arrive at
$$\mathbb{P}\Big(T_1^{\mathbf{p},\mathbf{q}} \lesssim \eta\sqrt{\frac{n\|\mathbf{p}-\mathbf{q}\|_2\log(r^{-1})}{m}}, \forall(\mathbf{p},\mathbf{q}) \in \mathcal{N}_{r,\delta_4}^{(2)}\Big) \geq 1-\exp(-2n\log(\frac{6}{r}))$$

Bias Term $T_2^{\mathbf{p},\mathbf{q}}$

Goal: We need precise expression of

$$T_2^{\mathbf{p},\mathbf{q}} = \Big|\eta \cdot \underbrace{\mathbb{E}\big[|\mathbf{a}_i^T\boldsymbol{\beta}_1|\mathbf{1}(i \in \mathbf{R}_{\mathbf{p},\mathbf{q}})\big]}_{I \text{ to be computed}} - \|\mathbf{p}-\mathbf{q}\|_2\Big|$$

when $\|\mathbf{p}-\mathbf{q}\|_2 \lesssim 1$

Parameterization of $(\mathbf{p},\mathbf{q})$: by $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$,

$$\mathbf{p} = u_1\boldsymbol{\beta}_1 + u_2\boldsymbol{\beta}_2\,, \quad \mathbf{q} = v_1\boldsymbol{\beta}_1 + u_2\boldsymbol{\beta}_2, \quad |u_1 - v_1| \lesssim 1$$

so

$$\{i \in \mathbf{R}_{\mathbf{p},\mathbf{q}}\} = \{sign(|u_1\mathbf{a}_i^T\boldsymbol{\beta}_1 + u_2\mathbf{a}_i^T\boldsymbol{\beta}_2| - 1) \neq sign(|v_1\mathbf{a}_i^T\boldsymbol{\beta}_1 + u_2\mathbf{a}_i^T\boldsymbol{\beta}_2| - 1)\}$$

and hence

$$I := \mathbb{E}_{g_1,g_2\sim N(0,1)}\big[|g_1|\mathbf{1}\big(sign(|u_1g_1 + u_2g_2| - 1) \neq sign(|v_1g_1 + u_2g_2| - 1)\big)\big]$$

Bias Term $T_2^{\mathbf{p},\mathbf{q}}$

by some discussion,

$$I = \sqrt{\frac{2}{\pi}} \int_0^\infty z \exp(-\frac{z^2}{2})(P_1 + P_2)\, dz$$

$$P_1 := \mathbb{P}\Big(\frac{\max\{1 - zu_1, -1 - zv_1\}}{u_2} < N(0,1) < \frac{1 - zv_1}{u_2} \Big)$$

$$P_2 := \mathbb{P}\Big(\frac{\max\{zu_1 - 1, zv_1 + 1\}}{u_2} < N(0,1) < \frac{1 + zu_1}{u_2} \Big)$$

Too complicated to have closed form !

Remedy: Ignore an imprecision of $o(\|\mathbf{p} - \mathbf{q}\|_2)$

$$\mathbb{E}\big[|\mathbf{a}_i^T \beta_1| \mathbf{1}(i \in \mathbf{R}_{\mathbf{p},\mathbf{q}})\big] = (*) + o(\|\mathbf{p} - \mathbf{q}\|_2)$$

This does not affect choosing η in view of

$$T_2^{\mathbf{p},\mathbf{q}} = \Big| \eta \cdot \mathbb{E}\big[|\mathbf{a}_i^T \beta_1| \mathbf{1}(i \in \mathbf{R}_{\mathbf{p},\mathbf{q}})\big] - \|\mathbf{p} - \mathbf{q}\|_2 \Big|$$

Bias Term $T_2^{\mathbf{p},\mathbf{q}}$

Example: the replacement

$$I \longrightarrow I' = \sqrt{\frac{2}{\pi}} \int_0^\infty z \exp(-\frac{z^2}{2})(P_1' + P_2')\, dz$$

$$P_1 \longrightarrow P_1' := \mathbb{P}\Big(\frac{1 - zu_1}{u_2} < N(0,1) < \frac{1 - zv_1}{u_2}\Big)$$

$$P_2 \longrightarrow P_2' := \mathbb{P}\Big(\frac{zv_1 + 1}{u_2} < N(0,1) < \frac{zu_1 + 1}{u_2}\Big)$$

induces difference bounded by $O(\exp(-\frac{1}{2\|\mathbf{p}-\mathbf{q}\|_2^2}))$

Result:

$$T_2^{\mathbf{p},\mathbf{q}} = \|\mathbf{p} - \mathbf{q}\|_2 \Big|1 + o(1) - \eta g(u_1, u_2)\Big|$$

$$g(a, b) := \sqrt{\frac{2}{\pi}} \exp\Big(-\frac{1}{2(a^2 + b^2)}\Big) \frac{a^2 + b^2(a^2 + b^2)}{(a^2 + b^2)^{5/2}}$$

Term 2

by centering Term 2 $= \eta |\langle \mathbf{h}_1(\mathbf{p},\mathbf{q}), \beta_2\rangle|$,

$$\eta |\langle \mathbf{h}_1(\mathbf{p},\mathbf{q}), \beta_2\rangle| = \eta \Big| \frac{1}{m} \sum_{i\in \mathbf{R}_{\mathbf{p},\mathbf{q}}} sign(\mathbf{a}_i^T\beta_1)\mathbf{a}_i^T\beta_2 \Big|$$

$$\leq \overbrace{\eta \Big| \frac{1}{m}\sum_{i=1}^m \big(sign(\mathbf{a}_i^T\beta_1)\mathbf{a}_i^T\beta_2 \mathbf{1}(i\in \mathbf{R}_{\mathbf{p},\mathbf{q}}) - \mathbb{E}\big[sign(\mathbf{a}_i^T\beta_1)\mathbf{a}_i^T\beta_2\mathbf{1}(i\in\mathbf{R}_{\mathbf{p},\mathbf{q}})\big]\big)\Big|}^{\text{concentration term } T_3^{\mathbf{p},\mathbf{q}}}$$

$$+ \underbrace{\eta \Big| \mathbb{E}\big[sign(\mathbf{a}_i^T\beta_1)\mathbf{a}_i^T\beta_2\mathbf{1}(i\in\mathbf{R}_{\mathbf{p},\mathbf{q}})\big]\Big|}_{\text{bias term } T_4^{\mathbf{p},\mathbf{q}}}$$

similar arguments yield

$$|T_3^{\mathbf{p},\mathbf{q}}| \lesssim \eta \sqrt{\frac{\|\mathbf{p}-\mathbf{q}\|_2 n \log(1/r)}{m}}, \quad \forall (\mathbf{p},\mathbf{q}) \in \mathcal{N}^{(2)}_{r,\delta_4}, \quad (w.h.p.)$$

$$T_4^{\mathbf{p},\mathbf{q}} = \eta \|\mathbf{p}-\mathbf{q}\|_2 \big| h(u_1,u_2) + o(1)\big|, \quad h(a,b) = \sqrt{\frac{2}{\pi}} \exp\Big(-\frac{1}{2(a^2+b^2)}\Big)\frac{ab(a^2+b^2-1)}{(a^2+b^2)^{5/2}}$$

Term 3 & Final Bound on Contraction Term

Term 3. It is centered and bounded by

$$\eta\|\mathbf{h}_1^{\perp}(\mathbf{p},\mathbf{q})\|_2 \lesssim \eta\sqrt{\frac{\|\mathbf{p}-\mathbf{q}\|_2 n\log(r^{-1})}{m}}, \quad \forall(\mathbf{p},\mathbf{q})\in\mathcal{N}_{r,\delta_4}^{(2)}, \quad (w.h.p.)$$

Final bound. for all $(\mathbf{p},\mathbf{q})\in\mathcal{N}_{r,\delta_4}^{(2)}$,

$$\|\mathbf{p}-\mathbf{q}-\eta\cdot\mathbf{h}_1(\mathbf{p},\mathbf{q})\|_2 \le T_1^{\mathbf{p},\mathbf{q}} + T_3^{\mathbf{p},\mathbf{q}} + T_2^{\mathbf{p},\mathbf{q}} + T_4^{\mathbf{p},\mathbf{q}} + \eta\|\mathbf{h}_1^{\perp}(\mathbf{p},\mathbf{q})\|_2$$

$$\le \underbrace{O\Big(\eta\sqrt{\frac{n\log(r^{-1})}{m}}\cdot\|\mathbf{p}-\mathbf{q}\|_2\Big)}_{\delta_2=\tilde{O}(n/m)} \qquad \blacktriangleright \text{ concentration terms}$$

$$+\|\mathbf{p}-\mathbf{q}\|_2\underbrace{\Big[\big|1-\eta g(u_1,u_2)\big|+\eta|h(u_1,u_2)|+o(1)\Big]}_{\substack{\text{choose } \eta \text{ such that } <1-\epsilon \text{ to get contraction} \\ \eta=\sqrt{\pi e/2}\text{ works well !}}} \qquad \blacktriangleright \text{ bias terms}$$

Approximation Term

Approximation Term

Goal: Show

$$\overbrace{\|\mathbf{h}(\mathbf{u},\mathbf{v})-\mathbf{h}(\mathbf{u}_1,\mathbf{v}_1)\|_2=\tilde{O}\Big(\frac{n}{m}\Big)}^{\delta_3=\tilde{O}(n/m)},\quad \overbrace{\forall \mathbf{u},\mathbf{v}\in\mathbb{A}^2_{1/2}}^{\text{remain infinite !}},\ \|\mathbf{u}-\mathbf{v}\|_2\le\delta_4$$

$$\text{where}\quad \underbrace{\mathbf{h}(\mathbf{u},\mathbf{v})=\frac{1}{2m}\sum_{i=1}^m\big(sign(|\mathbf{a}_i^T\mathbf{u}|-1)-sign(|\mathbf{a}_i^T\mathbf{v}|-1)\big)sign(\mathbf{a}_i^T\mathbf{u})\mathbf{a}_i}_{\text{no more than } |\mathbf{R}_{\mathbf{u},\mathbf{v}}|=|\{i\in[m]\,:\,sign(|\mathbf{a}_i^T\mathbf{u}|-1)\neq sign(|\mathbf{a}_i^T\mathbf{v}|-1)\}| \text{ contributors}}$$

Main Ideas:

1. cancellation between $\mathbf{h}(\mathbf{u},\mathbf{v})-\mathbf{h}(\mathbf{u}_1,\mathbf{v}_1)$
2. local binary embedding of S^{n-1} (Oymak and Recht, 2015)
3. phaseless local binary embedding of $\mathbb{A}_{1/2,2}$ (this work)

Local Binary Embedding of S^{n-1}

Binary embedding of $X \subset R^n$: a map $\mathcal{M} : X \to \{\pm 1\}^m$ such that certain relation between ℓ_2-distance $\|x_1 - x_2\|_2$ and the normalized hamming distance $\frac{1}{m} d_H(\mathcal{M}(x_1), \mathcal{M}(x_2))$ holds for all $x_1, x_2 \in X$.

'Local' binary embedding of $X \subset R^n$: a map $\mathcal{M} : X \to \{\pm 1\}^m$ such that certain relation between ℓ_2-distance $\|x_1 - x_2\|_2$ and the normalized hamming distance $\frac{1}{m} d_H(\mathcal{M}(x_1), \mathcal{M}(x_2))$ holds for all $x_1, x_2 \in X$ obeying $\|x_1 - x_2\|_2 \leq \delta$.

Lemma (Local binary embedding of S^{n-1} via $\mathcal{M}(\mathbf{u}) = sign(\mathbf{Au})$ (Oymak and Recht, 2015))

If $m \gtrsim n$, then for any $r = \tilde{\Omega}(\frac{n}{m})$, w.h.p., for all $\mathbf{u}, \mathbf{v} \in S^{n-1}$,

$$\underbrace{\|\mathbf{u} - \mathbf{v}\|_2 \leq r \implies \frac{d_H(sign(\mathbf{Au}), sign(\mathbf{Av}))}{m} = \frac{|\mathbf{L}_{\mathbf{u},\mathbf{v}}|}{m} = \tilde{O}(r)}_{\substack{\text{this is simply a consequence of Chernoff bound if considering a fixed pair } (\mathbf{u},\mathbf{v}) \\ \text{the key is uniformity}}}$$

Phaseless local binary embedding of $\mathbb{A}^2_{1/2}$

Lemma (Phaseless local binary embedding of $\mathbb{A}^2_{1/2}$ via $\mathcal{M}(\mathbf{u}) = sign(|\mathbf{Au}| - 1)$ (This work))

If $m \gtrsim n$, then for any $r = \tilde{\Omega}(\frac{n}{m})$, w.h.p., for all $\mathbf{u}, \mathbf{v} \in \mathbb{A}^2_{1/2}$,

$$dist(\mathbf{u}, \mathbf{v}) \le r \implies \frac{d_H(sign(|\mathbf{Au}| - 1), sign(|\mathbf{Av}| - 1))}{m} = \frac{|\mathbf{R}_{\mathbf{u},\mathbf{v}}|}{m} = \tilde{O}(r)$$

Trivial but insufficient bound

Sufficient Condition: There are at most $\tilde{O}(n)$ nonzero contributors to $\mathbf{h}(\mathbf{u},\mathbf{v}) - \mathbf{h}(\mathbf{u}_1,\mathbf{v}_1)$

Why Sufficient? a known uniform bound on sub-Gaussian projections (e.g., Dirksen and Mendelson (2021)) further yields

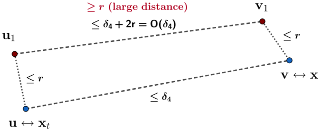
$$\|\mathbf{h}(\mathbf{u},\mathbf{v}) - \mathbf{h}(\mathbf{u}_1,\mathbf{v}_1)\|_2 = \tilde{O}(\frac{n}{m})$$

Note a trivial bound on the number of nontrivial contributors to $\mathbf{h}(\mathbf{u},\mathbf{v})$ and $\mathbf{h}(\mathbf{u}_1,\mathbf{v}_1)$

$$\overbrace{|\mathbf{R}_{\mathbf{u},\mathbf{v}}|}^{\text{bound on } \mathbf{h}(\mathbf{u},\mathbf{v})} + \overbrace{|\mathbf{R}_{\mathbf{u}_1,\mathbf{v}_1}|}^{\text{bound on } \mathbf{h}(\mathbf{u}_1,\mathbf{v}_1)} \underbrace{\leq m \cdot \tilde{O}(\delta_4)}_{\text{embedding of } \mathbb{A}^2_{1/2}} \stackrel{\delta_4 = \tilde{O}(1)}{=} \tilde{O}(m)$$

which is looser than the desired $\tilde{O}(n)$.

Rearrangement



$$\mathbf{h}(\mathbf{u}_1, \mathbf{v}_1) - \mathbf{h}(\mathbf{u}, \mathbf{v})$$

$$= \underbrace{\frac{1}{2m}\sum_{i=1}^{m}\left[sign(|\mathbf{a}_i^\top \mathbf{v}| - 1) - sign(|\mathbf{a}_i^\top \mathbf{v}_1| - 1)\right] sign(\mathbf{a}_i^\top \mathbf{u}_1)\mathbf{a}_i}_{\text{no more than } |\mathbf{R}_{\mathbf{v},\mathbf{v}_1}|}$$

$$+ \underbrace{\frac{1}{2m}\sum_{i=1}^{m}\left[sign(|\mathbf{a}_i^\top \mathbf{u}_1| - 1) - sign(|\mathbf{a}_i^\top \mathbf{u}| - 1)\right] sign(\mathbf{a}_i^\top \mathbf{u}_1)\mathbf{a}_i}_{\text{no more than } |\mathbf{R}_{\mathbf{u}_1,\mathbf{u}}|}$$

$$+ \underbrace{\frac{1}{2m}\sum_{i=1}^{m}\left[sign(\mathbf{a}_i^\top \mathbf{u}) - sign(\mathbf{a}_i^\top \mathbf{u}_1)\right]\left[sign(|\mathbf{a}_i^\top \mathbf{v}| - 1) - sign(|\mathbf{a}_i^\top \mathbf{u}| - 1)\right]\mathbf{a}_i,}_{\text{no more than } |\mathbf{L}_{\mathbf{u},\mathbf{u}_1}|}$$

Final Bound

The number of nonzero contributors to

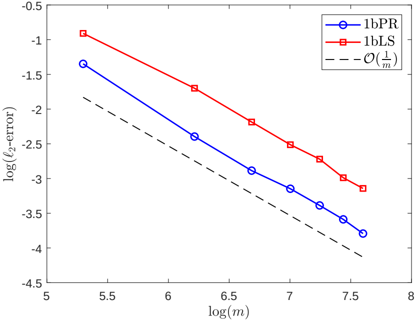
$$\mathbf{h}(\mathbf{u}_1, \mathbf{v}_1) - \mathbf{h}(\mathbf{u}, \mathbf{v})$$

is bounded by

$$|\mathbf{R}_{\mathbf{v},\mathbf{v}_1}| + |\mathbf{R}_{\mathbf{u}_1,\mathbf{u}}| + |\mathbf{L}_{\mathbf{u},\mathbf{u}_1}| = \tilde{O}(mr) = \tilde{O}(n)$$

Experiments

Synthetic data: $O(m^{-1})$ decay rate



blue curve: recover $\mathbf{x} \in S^{29}$ from $\mathbf{y} = sign(|\mathbf{Ax}| - 1)$

red curve: recover $\mathbf{x} \in S^{29}$ from $\mathbf{y} = sign(\mathbf{Ax})$

Natural image

(a) Original image: Milky Way Galaxy.

(b) Recovered image after spectral initialization: relative error = 0.270, PSNR = 25.14.

(c) Recovered image after gradient descent : relative error = 0.029, PSNR = 44.65.

Figure: *Recovering the* $1080 \times 1980 \times 3$ *Milky Way Galaxy image from phaseless bits produced by CDP with* $L = 64$ *random patterns.*

Conclusion

Concluding Remarks

1bPR. recovery $1/2 \le \|\mathbf{x}\|_2 \le 2$ from $\mathbf{y} = sign(|\mathbf{Ax}| - 1)$

What we show? Under $m \gtrsim n$, gradient descent w.r.t. ReLU loss with a spectral initialization attains optimal error

$$\sup_{\mathbf{x}\in\mathbb{A}^2_{1/2}} dist(\hat{\mathbf{x}},\mathbf{x}) = \tilde{O}(\frac{n}{m})$$

$\mathbf{x}$ is k-sparse? (Chen and Yuan, 2024) Under $m = \tilde{O}(k^2)$, thresholded gradient descent w.r.t. ReLU loss with a spectral initialization attains $dist(\hat{\mathbf{x}},\mathbf{x}) = \tilde{O}(\frac{k}{m})$ — again, this's information-theoretic optimal

Open questions.

- random initialization;
- complex-valued signal;
- precise analysis ...

“One-Bit Phase Retrieval: Optimal Rates and Efficient Algorithms”

Available on arXiv: https://arxiv.org/abs/2405.04733

Thank You!

Questions?

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