Efficient and Optimal Quantized Compressed Sensing

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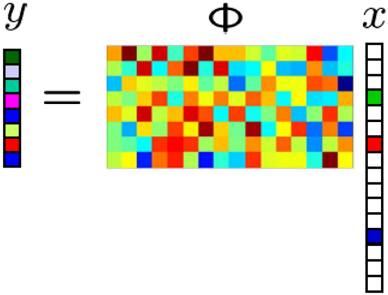
Outline

(Linear) Compressed Sensing

- Goal: recover *structured* $\mathbf{x} \in \mathbb{R}^n$ from $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{y} = \mathbf{A}\mathbf{x}$
- $\mathbf{A} = [\mathbf{a}_1, \cdots, \mathbf{a}_m]^\top$, then we observe $y_i = \mathbf{a}_i^\top \mathbf{x}, \quad i = 1, \cdots, m$
- Result: k-sparse $\mathbf{x}$ can be exactly recovered from $O(k \log(\frac{en}{k}))$ Gaussian measurements via constrained ℓ_1-norm minimization

$$\hat{\mathbf{x}}_{\mathrm{bp}} = \arg\min \|\mathbf{u}\|_1, \quad \text{s.t. } \mathbf{A}\mathbf{u} = \mathbf{y} \tag{1}$$



Nonlinear Compressed Sensing

- Goal: recover structured $\mathbf{x} \in \mathbb{R}^n$ from $\mathbf{A} \in \mathbb{R}^{m\times n}$ from $y_i = f_i(\mathbf{a}_i^\top \mathbf{x}),\ i = 1,\cdots,m$
- Result: Under Gaussian matrix and fairly mild condition on $\{f_i\}_{i=1}^m$, we can **ignore the nonlinearity** and use G-Lasso

$$\hat{\mathbf{x}}_{\text{GLasso}} = \arg\min \frac{1}{2m}\|\mathbf{y}-\mathbf{A}\mathbf{u}\|_2^2 + \lambda\|\mathbf{u}\|_1 \tag{2}$$

to recover k-sparse $\mathbf{x}$ to ℓ_2 error [PV16][1]

$$\|\hat{\mathbf{x}}_{\text{GLasso}} - \mathbf{x}\|_2 = \tilde{O}\Big(\sqrt{\frac{k}{m}}\Big)$$

- In general, we cannot do better without knowing f_i !
  Just think of noisy linear regression $y = \mathbf{A}\mathbf{x} + \epsilon$ with $\epsilon \sim \mathcal{N}(0,\sigma^2\mathbf{I}_m)$

[1] The generalized lasso with non-linear observations. Y. Plan & R. Vershynin, 2016, TIT.

Quantizer

- An L-level quantizer Q which quantizes $q \in \mathbb{R}$ to

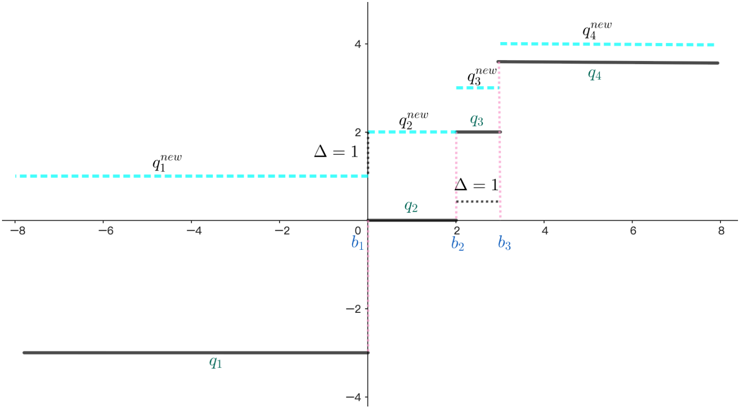
$$Q(q) = \begin{cases} q_1, & \text{if } q < b_1 \\ q_2, & \text{if } b_1 \le q < b_2 \\ \cdots \\ q_{L-1}, & \text{if } b_{L-2} \le q < b_{L-1} \\ q_L, & \text{if } q \ge b_{L-1} \end{cases} \tag{3}$$

- for some quantization thresholds $b_1 < b_2 < \cdots < b_{L-1}$
- and some quantized values $q_1 < q_2 < \cdots < q_L$.
- Resolution: If $L \ge 3$, we define

$$\Delta := \min_{j=1,\cdots,L-2} |b_{j+1} - b_j|; \tag{4}$$

If $L = 2$, we define $\Delta := 2$ (just a convention).

Quantizer



The values $(q_i)_{i=1}^{L-1}$ are not important, so we could assume

$$q_{i+1} = q_i + \Delta, \quad i = 1, 2, \cdots, L-1. \tag{5}$$

Quantized Compressed Sensing

An important instance of nonlinear CS:

- Goal of Quantized CS: recover *structured* signal $\mathbf{x} \in \mathbb{R}^n$ from

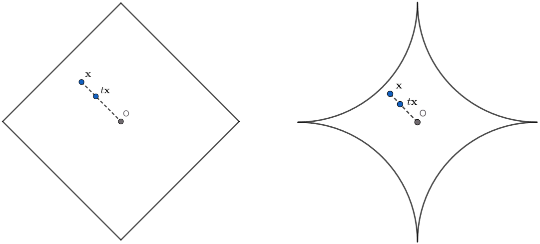
$$\mathbf{y} = Q(\mathbf{A}\mathbf{x} - \boldsymbol{\tau}) = \begin{bmatrix} Q(\mathbf{a}_1^\top \mathbf{x} - \tau_1) \\ Q(\mathbf{a}_2^\top \mathbf{x} - \tau_2) \\ \vdots \\ Q(\mathbf{a}_m^\top \mathbf{x} - \tau_m) \end{bmatrix}$$

- $\mathbf{A} \in \mathbb{R}^{m \times n}$: we focus on *sub-Gaussian* matrix
- $\boldsymbol{\tau} \in \mathbb{R}^m$: dithering noise helps reconstruction [JR72][2]
- we focus on $\boldsymbol{\tau} \sim \mathscr{U}[-\Lambda, \Lambda]^m$ **independent of A**
- $\Lambda = 0$ reduces to the non-dithered case

[2] The application of dither to the quantization of speech signals. N. Jayant, L. Rabiner, 1972.

Signal Structure

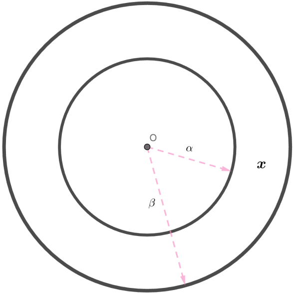
- $\mathbf{x} \in \mathcal{K}$ for star-shaped set $\mathcal{K}$ (*Definition*: $\forall \mathbf{u} \in \mathcal{K}$, $t\mathbf{u} \in \mathcal{K}$ for any $t \in [0,1]$)



- Examples:
  - k-sparse signals $\Sigma_k^n = \{\mathbf{u} \in \mathbb{R}^n : \|\mathbf{u}\|_0 \le k\}$;
  - low-rank matrices $M_{\bar{r}}^{n_1,n_2} = \{\mathbf{M} \in \mathbb{R}^{n_1 \times n_2} : \text{rank}(\mathbf{M}) \le \bar{r}\}$;
  - effectively sparse signals $\sqrt{k}\mathbb{B}_1^n = \{\mathbf{u} \in \mathbb{R}^n : \|\mathbf{u}\|_1 \le \sqrt{k}\}$.

Signal Norm

- $\mathbf{x} \in \mathbb{A}_{\alpha}^{\beta} = \{\mathbf{u} \in \mathbb{R}^n : \alpha \leq \|\mathbf{u}\|_2 \leq \beta\}.$



Signal space: Taken collectively, we consider the recovery of the signals in

$$\mathscr{X} := \mathscr{K} \cap \mathbb{A}_{\alpha}^{\beta} \tag{6}$$

One-Bit Compressed Sensing (1bCS)

Problem setup

- Recover $\mathbf{x}$ from $\mathbf{y} = \text{sign}(\mathbf{A}\mathbf{x}) = (\text{sign}(\mathbf{a}_1^\top \mathbf{x}), \cdots, \text{sign}(\mathbf{a}_m^\top \mathbf{x}))^\top$, with $\mathbf{A} \sim \mathcal{N}^{m\times n}(0,1)$
- $\mathbf{x} \in \mathcal{K} \cap \mathbb{S}^{n-1} \longrightarrow$ we cannot distinguish $\mathbf{x}$ and $2\mathbf{x}$

Optimal rate

- Hamming distance $d_H(\mathbf{u},\mathbf{v}) = \sum_{i=1}^m \mathbf{1}(u_i \neq v_i)$
- Hamming distance minimization (HDM):

$$\hat{\mathbf{x}}_{hdm} = \arg \min_{\mathbf{u}\in\mathcal{K}\cap\mathbb{S}^{n-1}} d_H(\text{sign}(\mathbf{A}\mathbf{u}), \mathbf{y}) \tag{7}$$

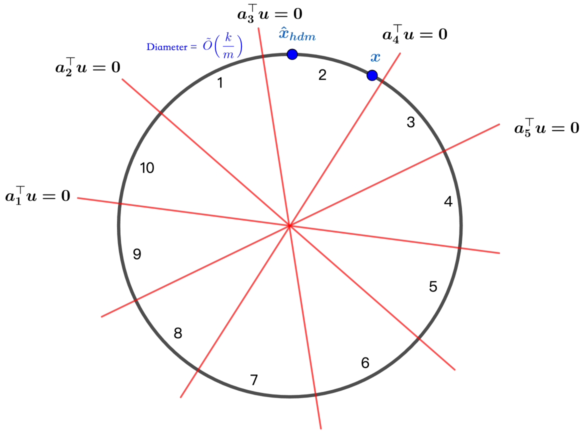
- $\mathcal{K} = \Sigma_k^n$: $\|\hat{\mathbf{x}}_{hdm} - \mathbf{x}\|_2 = \tilde{O}(\frac{k}{m})$ [JLBB13][3] (Optimal rate)
- This is sharper than $\Theta(\sqrt{k/m})$ for noisy regression
- $\mathcal{K} = \sqrt{k}\mathbb{B}_1^n$: $\|\hat{\mathbf{x}}_{hdm} - \mathbf{x}\|_2 = \tilde{O}\big((\frac{k}{m})^{1/3}\big)$ [OR15][4] (Fastest rate)

[3] Robust 1-Bit Compressive Sensing via Binary Stable Embeddings of Sparse Vectors. L. Jacques, J. Laska, P. T. Boufounos; R. Baraniuk, 13 TIT.

[4] Near-optimal bounds for binary embeddings of arbitrary sets. S. Oymak & B. Recht, Arxiv, 2015.

Hyperplane Tessellation - 1bCS

1bCS $\Longleftrightarrow$ hyperplane tessellation of a subset of $\mathbb{S}^{n-1}$ [PV14][5]

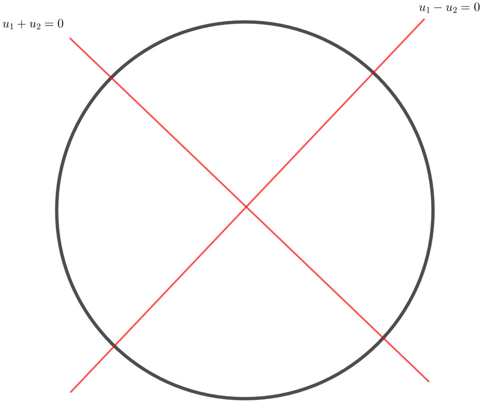


[5] Dimension reduction by random hyperplane tessellations, Y. Plan & R. Vershynin, 2014 Discrete & Computational Geometry

Geometry of 1bCS - Gaussian Hyperplane Tessellation

Go beyond Gaussian design? [ALPV14];[6]

$\{-1, 1\}$-valued $\mathbf{A}$ (e.g., Bernoulli design) does not work:



[6] One-bit compressed sensing with non-Gaussian measurements, A. Ai, A. Lapanowski, Y. Plan, R. Vershynin, Linear Algebra and its Applications, 2014.

Efficient Algorithms for 1bCS

algorithm	error rate	signal space	uniformity
Linear Program [PV13][7]	$\tilde{O}((\frac{k}{m})^{1/5})$	$\sqrt{k}\mathbb{B}_1^n \cap \mathbb{S}^{n-1}$	✓
Convex Relaxation [PV12][8]	$\tilde{O}((\frac{k}{m})^{1/4})$	$\sqrt{k}\mathbb{B}_1^n \cap \mathbb{S}^{n-1}$	✗
Generalized Lasso [PV16]	$\tilde{O}((\frac{k}{m})^{1/2})$	$\Sigma_k^n \cap \mathbb{S}^{n-1}$	✗
PBP [PVY17][9]	$\tilde{O}((\frac{k}{m})^{1/2})$	$\Sigma_k^n \cap \mathbb{S}^{n-1}$	✗
Adaboost [CKLG22][10]	$\tilde{O}((\frac{k}{m})^{1/3})$	$\sqrt{k}\mathbb{B}_1^n \cap \mathbb{S}^{n-1}$	✓
NBIHT [MM24][11]	$\tilde{O}(\frac{k}{m})$	$\Sigma_k^n \cap \mathbb{S}^{n-1}$	✓
PGD (our work)	$\tilde{O}(\frac{\bar{r}(n_1+n_2)}{m})$	$M_{\bar{r}}^{n_1,n_2} \cap \{\lVert \mathbf{U} \rVert_F = 1\}$	✓
PGD (our work)	$\tilde{O}((\frac{k}{m})^{1/3})$	$\sqrt{k}\mathbb{B}_1^n \cap \mathbb{S}^{n-1}$	✓

[7] One-bit compressed sensing by linear programming, Y. Plan & R. Vershynin, CPAM, 2013.

[8] Robust 1-bit compressed sensing and sparse logistic regression: A convex programming approach, Y. Plan & R. Vershynin, 12 TIT.

[9] High-dimensional estimation with geometric constraints, Y. Plan, R. Vershynin & E. Yudovina, 17 inf. inference.

[10] Adaboost and robust one-bit compressed sensing, G. Chinot, F. Kuchelmeister, M. Löffler, S. Geer, 22 MSL.

[11] Binary iterative hard thresholding converges with optimal number of measurements for 1-bit compressed sensing, N. Matsumoto & A. Mazumdar, 24 JACM

Dithered One-Bit Compressed Sensing (D1bCS)

Downsides of 1bCS

- hard to go beyond Gaussian design;
- unable to recover signal norm (see a fix by Gaussian dither [KSW16][12])

D1bCS Problem Setup

- using *uniform dither* $\boldsymbol{\tau} \sim \mathscr{U}[-\lambda, \lambda]^m$ addresses both issues [DM21],[13] [TR20][14]
- Model: recover $\mathbf{x} \in \mathcal{K} \cap \mathbb{B}_2^n$ from

$$\mathbf{y} = \operatorname{sign}(\mathbf{Ax} - \boldsymbol{\tau})$$

under *sub-Gaussian* $\mathbf{A}$

[12] One-bit compressive sensing with norm estimation, K. Knudson, R. Saab, R. Ward, 16 TIT.

[13] Non-Gaussian hyperplane tessellations and robust one-bit compressed sensing, S. Dirksen & S. Mendelson, 2021 JEMS

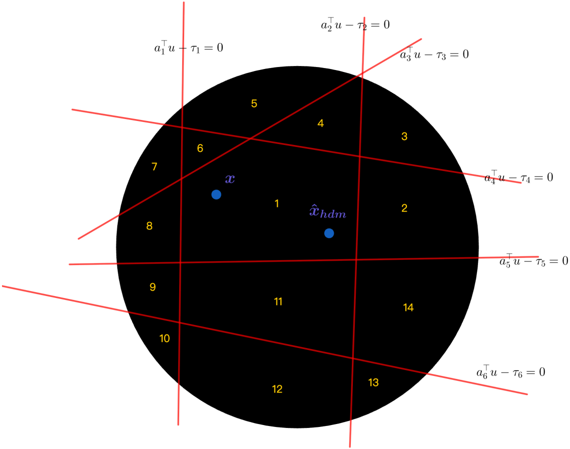
[14] The generalized lasso for sub-gaussian measurements with dithered quantization, C. Thrampoulidis & A. S. Rawat, 20 TIT.

Geometric of D1bCS - Non-Gaussian Hyperplane Tessellation

- Theorem 1.9 in [DM21]: HDM

$$\hat{\mathbf{x}}_{hdm} = \arg \min_{\mathbf{u}\in\mathcal{K}\cap\mathbb{B}_2^n} d_H\big(\mathrm{sign}(\mathbf{A}\mathbf{u}-\boldsymbol{\tau}),\mathbf{y}\big) \qquad (8)$$

achieves the optimal rate $\tilde{O}(\frac{k}{m})$ if $\mathcal{K}=\Sigma_k^n$, and the fastest rate $\tilde{O}((\frac{k}{m})^{1/3})$ if $\mathcal{K}=\sqrt{k}\mathbb{B}_1^n$



Efficient Algorithms for D1bCS

algorithm	error rate	signal space	uniformity
Convex Relaxation [DM21]	$\tilde{O}((\frac{k}{m})^{1/4})$	$\sqrt{k}\mathbb{B}_1^n \cap \mathbb{B}_2^n$	✓
Con. Rel. (one-sided ℓ_1) [JMPS21][15]	$\tilde{O}((\frac{k}{m})^{1/3})$	$\sqrt{k}\mathbb{B}_1^n \cap \mathbb{B}_2^n$	✓
Generalized Lasso [TR20]	$\tilde{O}((\frac{k}{m})^{1/2})$	$\Sigma_k^n \cap \mathbb{B}_2^n$	✗
PGD (our work)	$\tilde{O}(\frac{k}{m})$	$\Sigma_k^n \cap \mathbb{B}_2^n$	✓
PGD (our work)	$\tilde{O}((\frac{k}{m})^{1/3})$	$\sqrt{k}\mathbb{B}_1^n \cap \mathbb{B}_2^n$	✓

[15] Quantized Compressed Sensing by Rectified Linear Units, H. C. Jung, J. Maly, L. Palzer, A. Stollenwerk, 21 TIT.

Dithered Multi-Bit Compressed Sensing (DMbCS)

- Another benefit of dithering: *generalization to multi-bit sensing*
- We consider the uniform quantizer

$$Q_\delta(a) = \delta\left(\left\lfloor \frac{a}{\delta} \right\rfloor + \frac{1}{2}\right) = \begin{cases} \cdots \\ -\frac{3\delta}{2}, & \text{if } a \in (-2\delta, -\delta) \\ -\frac{\delta}{2}, & \text{if } a \in (-\delta, 0) \\ \frac{\delta}{2}, & \text{if } a \in (0, \delta) \\ \frac{3\delta}{2}, & \text{if } a \in (\delta, 2\delta) \\ \cdots \end{cases} . \tag{9}$$

- **QCS with Dithered Uniform Quantizer**: Under $\boldsymbol{\tau} \sim \mathscr{U}([-\frac{\delta}{2}, \frac{\delta}{2}]^m)$ and *sub-Gaussian* **A**, we can accurately recover structured signals $\mathbf{x} \in \mathcal{K} \cap \mathbb{B}_2^n$ from [TR20], [XJ20][16]

$$\mathbf{y} = Q_\delta(\mathbf{Ax} - \boldsymbol{\tau}). \tag{10}$$

[16] Quantized compressive sensing with rip matrices: The benefit of dithering, C. Xu & L. Jacques, 2020 Inf. inference.

Dithered Multi-Bit Compressed Sensing (DMbCS)

- Q_δ does not immediately sample finite bits
- For some even integer $L \geq 4$, we consider Q_δ with *saturation* when $\{a \in \mathbb{R} : |a| \geq \frac{L\delta}{2}\}$:

$$Q_{\delta,L}(a) = Q_\delta(a) \cdot \mathbb{1}\left(|a| < \frac{L\delta}{2}\right) + \frac{(L-1)\delta}{2}\mathbb{1}\left(a \geq \frac{L\delta}{2}\right) + \frac{(1-L)\delta}{2}\mathbb{1}\left(a \leq \frac{-L\delta}{2}\right) \tag{11}$$

- **DMbCS**: recover $\mathbf{x} \in \mathcal{K} \cap \mathbb{B}_2^n$ from $\mathbf{y} = Q_{\delta,L}(\mathbf{Ax} - \boldsymbol{\tau})$
- $\mathcal{K} = \Sigma_k^n$: optimal rate $\Omega(\frac{k}{mL})$ [BJKS15][17]
- $\mathcal{K} = \sqrt{k}\mathbb{B}_1^n$: the fastest rate $\tilde{O}((\frac{k}{mL})^{1/3})$ [JMPS21]
- Additional challenge - **nail down the role of quantization level** L

[17] Quantization and Compressive Sensing, P. T. Boufounos, L. Jacques, F. Krahmer, R. Saab, 2015.

Q_δ & $Q_{\delta,L}$ with Saturation

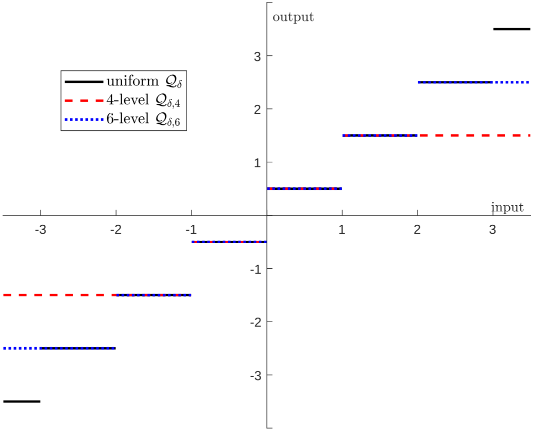
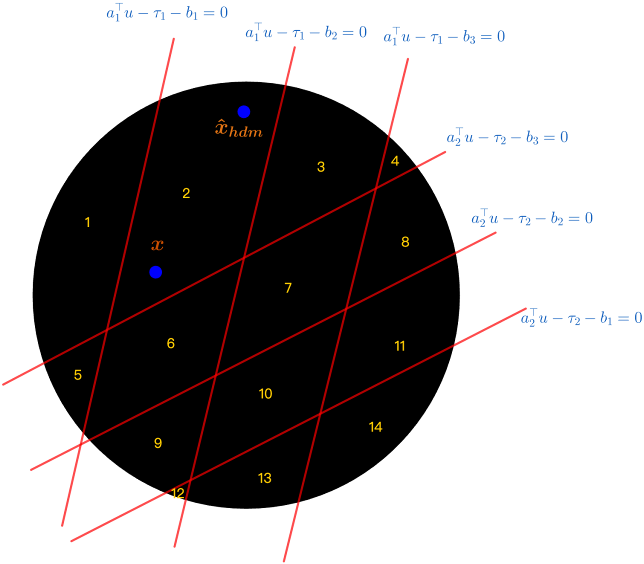


Figure 1: The uniform quantizer Q_δ and its saturated versions $Q_{\delta,4}$ and $Q_{\delta,6}$ under $\delta = 1$.

Geometric of DMbCS - Parallel Non-Gaussian Hyperplanes

$L = 4$



Efficient Algorithms for DMbCS

algorithm	error rate	signal space	uniformity
Con. Rel. (one-sided ℓ_1) [JMPS21]	$\tilde{O}\big((\frac{k}{mL})^{1/3}\big)$	$\sqrt{k}\mathbb{B}_1^n \cap \mathbb{B}_2^n$	✓, $Q_{\delta,L}$
Generalized Lasso [TR20]	$\tilde{O}\big(\frac{1}{L}(\frac{k}{m})^{1/2}\big)$	$\Sigma_k^n \cap \mathbb{B}_2^n$	✗, Q_δ
PBP [XJ20]	$\tilde{O}\big((1+\delta)(\frac{k}{m})^{1/2}\big)$	$\Sigma_k^n \cap \mathbb{B}_2^n$	✓, Q_δ
PGD (our work)	$\tilde{O}\big(\frac{k}{mL}\big)$	$\Sigma_k^n \cap \mathbb{B}_2^n$	✓, $Q_{\delta,L}$
PGD (our work)	$\tilde{O}\big((\frac{k}{mL})^{1/3}\big)$	$\sqrt{k}\mathbb{B}_1^n \cap \mathbb{B}_2^n$	✓, $Q_{\delta,L}$

Complexity of Arbitrary Set

How to capture complexity of an arbitrary set?

$$\omega(\mathscr{U}) = \mathbb{E}_{\mathbf{g}\sim\mathcal{N}(0,\mathbf{I}_n)} \sup_{\mathbf{u}\in\mathscr{U}} \langle \mathbf{g},\mathbf{u}\rangle\text{: Gaussian width}$$

$$\mathscr{N}(\mathscr{U}, r)\text{: covering number at scale } r$$

$$\log\mathscr{N}(\mathscr{U}, r)\text{: metric entropy at scale } r$$

Examples:

- $\omega(\mathbb{B}_2^n) \asymp \sqrt{n}$ and $\log\mathscr{N}(\mathbb{B}_2^n, r) \le n\log(\frac{C}{r})$
- $\omega(\Sigma_k^n \cap \mathbb{B}_2^n) \asymp \sqrt{k\log(\frac{en}{k})}$ and $\log\mathscr{N}(\Sigma_k^n \cap \mathbb{B}_2^n, r) \lesssim k\log(\frac{en}{rk})$
- $\omega(\sqrt{k}\mathbb{B}_1^n \cap \mathbb{B}_2^n) \asymp \sqrt{k\log(\frac{en}{k})}$ and $\log\mathscr{N}(\sqrt{k}\mathbb{B}_1^n \cap \mathbb{B}_2^n, r) \lesssim \frac{k}{r^2}\log(\frac{en}{rk})$

Separation

Separation Probability:

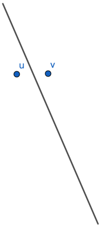
$$\mathsf{P}_{\mathbf{u},\mathbf{v}} := \mathbb{P}\left(Q(\mathbf{a}_i^\top \mathbf{u} - \tau_i) \neq Q(\mathbf{a}_i^\top \mathbf{v} - \tau_i)\right)$$

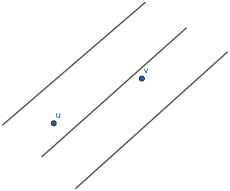
Separation Set:

$$\mathbf{R}_{\mathbf{u},\mathbf{v}} = \left\{i \in [m] : Q(\mathbf{a}_i^\top \mathbf{u} - \tau_i) \neq Q(\mathbf{a}_i^\top \mathbf{v} - \tau_i)\right\}$$

Separation Event:

$$E_{\mathbf{u},\mathbf{v}}^{(i)} := \{i \in \mathbf{R}_{\mathbf{u},\mathbf{v}}\}$$





Our Result - HDM Bound

HDM Bound: Under

- sub-Gaussian $\mathbf{A}$,
- a small-ball probability,
- a separation probability estimate,

(Informal) Theorem 1: Performance bound for the infeasible program - HDM

For small $r > 0$ and $r' \asymp \frac{r}{\log^{1/2}(r^{-1})}$, if

$$m \geq C_2(\Delta \vee \Lambda)\left(\frac{\omega^2(\mathcal{K}_{(r')})}{r^3} + \frac{\log \mathcal{N}(\mathcal{X}, r')}{r}\right) \tag{12}$$

then $\|\hat{\mathbf{x}}_{\text{hdm}} - \mathbf{x}\|_2 \leq 2r$.

- $\mathcal{K}_{(\phi)} := (\mathcal{K} - \mathcal{K}) \cap \phi \mathbb{B}_2^n$

Our Main Result - PGD Bound

PGD Bound: Under

- sub-Gaussian $\mathbf{A}$,
- small-ball probability,
- separation probability estimate,
- and **additionally a number of moment bounds** (that convey a weak type of independence between the sensing vector marginals and the separation event),

(Informal) Theorem 2: Performance bound for the efficient algorithm PGD

For small $r > 0$, if

$$m \geq C_2(\Delta \vee \Lambda)\left(\frac{\omega^2(\mathcal{K}_{(r)})}{r^3} + \frac{\log \mathcal{N}(\mathcal{X}, r/2)}{r}\right) \tag{13}$$

then $\|\hat{\mathbf{x}}_{\text{pgd}} - \mathbf{x}\|_2 = \tilde{O}(r)$.

Takeaway: Under some assumptions, PGD achieves the same rates as HDM.

Hamming Distance Minimization

QCS — General context

- recover structured signals $\mathbf{x} \in \mathscr{X} = \mathscr{K} \cap \mathbb{A}_{\alpha}^{\beta}$ from $\mathbf{y} = Q(\mathbf{A}\mathbf{x} - \boldsymbol{\tau})$ under sub-Gaussian $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{\tau} \sim \mathscr{U}[-\Lambda, \Lambda]^m$
- Q: L-level quantizer with resolution Δ

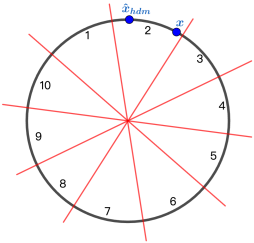
Hamming distance minimization (HDM)

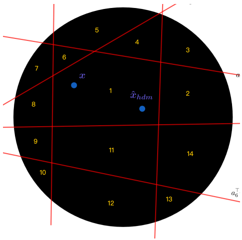
- HDM:

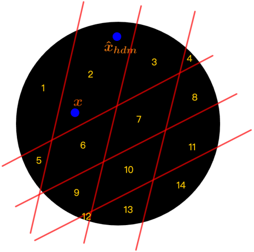
$$\hat{\mathbf{x}}_{\mathrm{hdm}} = \arg\min_{\mathbf{u} \in \mathscr{X}} d_H\big(Q(\mathbf{A}\mathbf{u} - \boldsymbol{\tau}), \mathbf{y}\big), \tag{14}$$

- This is the best possible program in the noiseless case

Hyperplane Tessellation







Assumptions 1 - Sub-Gaussian Design

- We first unify the HDM performance bounds for specific models—[JLBB13] and [OR15] for 1bCS; [DM21] for D1bCS—under generic assumptions

Assumption 1: Sub-Gaussian Design

The sensing vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_m$ are i.i.d. isotropic sub-Gaussian, i.e., they satisfy $\mathbb{E}(\mathbf{a}_i \mathbf{a}_i^\top) = \mathbf{I}_n$ and $\|\mathbf{a}_i\|_{\psi_2} \le C_0$ for some absolute constant C_0.

Assumptions 2, 3 - Small-Ball & Separation Probabilities

Assumption 2: Small-ball probability

For any $\mathbf{u} \in \mathbb{A}_{\alpha}^{\beta}$,

$$\mathbb{P}\left(\min_{j\in[L-1]} |\mathbf{a}_i^\top \mathbf{u} - \tau_i - b_j| \le t\right) \lesssim \frac{t}{\Delta \vee \Lambda}, \quad \forall t > 0.$$

Assumption 3: Separation probability

For any $\mathbf{u}, \mathbf{v} \in \mathbb{A}_{\alpha}^{\beta}$,

$$\mathsf{P}_{\mathbf{u},\mathbf{v}} \asymp \min\left\{\frac{\|\mathbf{u}-\mathbf{v}\|_2}{\Delta \vee \Lambda}, 1\right\}$$

Local Hyperplane Tessellation (Quantized Embedding Property)

Theorem 1: Quantized Embedding Property

Let $\mathscr{W}$ be a set contained in $\mathbb{A}_{\alpha}^{\beta}$, for $\epsilon \in (0, c_0(\Delta \vee \Delta))$, we let $\epsilon' = \frac{c_1 \epsilon}{\log^{1/2}(\frac{\Delta \vee \Lambda}{\epsilon})}$ and suppose

$$m \geq C_2(\Delta \vee \Lambda)\left(\frac{\omega^2(\mathscr{W}_{(3\epsilon'/2)})}{\epsilon^3} + \frac{\log \mathscr{N}(\mathscr{W}, \epsilon')}{\epsilon}\right). \quad (15)$$

- **(small to small)** Under Assumptions 1–2, with probability at least $1 - 2\exp(-\frac{c_3 m\epsilon}{\Delta \vee \Lambda})$, for any $\mathbf{u}, \mathbf{v} \in \mathscr{W}$

$$\|\mathbf{u} - \mathbf{v}\|_2 \leq \frac{\epsilon'}{2} \implies \underbrace{d_H\big(Q(\mathbf{A}\mathbf{u} - \boldsymbol{\tau}), Q(\mathbf{A}\mathbf{v} - \boldsymbol{\tau})\big)}_{:=|\mathbf{R}_{\mathbf{u},\mathbf{v}}|} \leq \frac{C_4 m\epsilon}{\Delta \vee \Lambda}.$$

- **(large to large)** Under Assumptions 1–3, with probability at least $1 - \exp(-\frac{c_5 m\epsilon}{\Delta \vee \Lambda})$, for any $\mathbf{u}, \mathbf{v} \in \mathscr{W}$

$$\|\mathbf{u} - \mathbf{v}\|_2 \geq 2\epsilon \implies d_H\big(Q(\mathbf{A}\mathbf{u} - \boldsymbol{\tau}), Q(\mathbf{A}\mathbf{v} - \boldsymbol{\tau})\big) \geq c_6\, m \min\left\{\frac{\|\mathbf{u} - \mathbf{v}\|_2}{\Delta \vee \Lambda}, 1\right\}.$$

$$d_H\big(Q(\mathbf{A}\hat{\mathbf{x}}_{\rm hdm} - \boldsymbol{\tau}), Q(\mathbf{A}\mathbf{x} - \boldsymbol{\tau})\big) = 0 \implies \|\hat{\mathbf{x}}_{\rm hdm} - \mathbf{x}\|_2 \leq 2\epsilon$$

Takeaway and Computational Problem

Takeaway:

Any sub-Gaussian QCS systems possessing a *separation probability proportional to ℓ_2 distance* and *a small-ball probability* achieves optimal sparse recovery rate $\tilde{O}(k/m)$

- Techniques: Similar to [OR15], [DM21]

However, $\hat{\mathbf{x}}_{\mathrm{hdm}}$ cannot be computed efficiently:

Can we devise an efficient algorithm to achieve error rates of $\hat{\mathbf{x}}_{\mathrm{hdm}}$?

HDM is Intractable

$$\hat{\mathbf{x}}_{\text{hdm}} = \arg\min_{\mathbf{u}\in\mathcal{X}} d_H\big(Q(\mathbf{A}\mathbf{u}-\boldsymbol{\tau}),\mathbf{y}\big)$$

- The intractability of $\hat{\mathbf{x}}_{\text{hdm}}$ comes from:

1. the discrete loss $d_H(Q(\mathbf{A}\mathbf{u}-\boldsymbol{\tau}),\mathbf{y}) = \sum_{i=1}^m \mathbf{1}\big(Q(\mathbf{a}_i^\top\mathbf{u}-\tau_i)\neq y_i\big)$
2. the typically nonconvex constraint $\mathbf{x}\in\mathcal{K}\cap\mathbb{A}_\alpha^\beta$

One-sided ℓ_1 loss

- Construct a (slightly different) hamming distance loss:
- Observing $y_i = Q(\mathbf{a}_i^\top \mathbf{x} - \tau_i)$ amounts to observing $L-1$ binary measurements

$$y_{ij} = \text{sign}(\mathbf{a}_i^\top \mathbf{x} - \tau_i - b_j), \quad j = 1, \cdots, L-1. \tag{16}$$

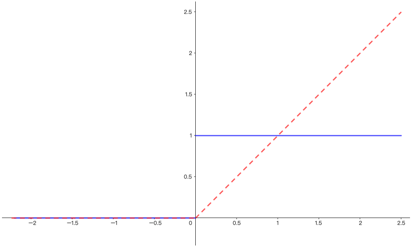
- With overall $(L-1)m$ binary measurements, we construct

$$\mathcal{L}(\mathbf{u}) := \frac{\Delta}{m} \sum_{i=1}^{m} \sum_{j=1}^{L-1} \mathbb{1}\left(\text{sign}(\mathbf{a}_i^\top \mathbf{u} - \tau_i - b_j) \neq y_{i,j}\right) \tag{17}$$

$$= \frac{\Delta}{m} \sum_{i=1}^{m} \sum_{j=1}^{L-1} \mathbb{1}\left(-y_{ij}(\mathbf{a}_i^\top \mathbf{u} - \tau_i - b_j) \geq 0\right), \tag{18}$$

One-sided ℓ_1 loss

- Common idea: to relax $\mathbf{1}(a>0)$ to the one-sided ℓ_1 loss $\max\{a,0\} = \frac{a+|a|}{2}$



- Then we obtain

$$\mathcal{L}_1(\mathbf{u}) = \frac{\Delta}{2m}\sum_{i=1}^{m}\sum_{j=1}^{L-1}\left[|\mathbf{a}_i^\top\mathbf{u} - \tau_i - b_j| - y_{ij}(\mathbf{a}_i^\top\mathbf{u} - \tau_i - b_j)\right].$$

with (sub-)gradient of $\mathcal{L}_1(\mathbf{u})$

$$\partial\mathcal{L}_1(\mathbf{u}) = \frac{\Delta}{2m}\sum_{i=1}^{m}\sum_{j=1}^{L-1}\left(\text{sign}(\mathbf{a}_i^\top\mathbf{u} - \tau_i - b_j) - y_{ij}\right)\mathbf{a}_i = \frac{1}{m}\sum_{i=1}^{m}\left(Q(\mathbf{a}_i^\top\mathbf{u} - \tau_i) - Q(\mathbf{a}_i^\top\mathbf{x} - \tau_i)\right)\mathbf{a}_i,$$

$$\uparrow q_{i+1} = q_i + \Delta$$

Nonconvex constraint

- How to deal with the nonconvex constraint?
- $\min_{\mathbf{u}\in\mathcal{X}} \mathcal{L}(\mathbf{u})$ is intractable for nonconvex $\mathcal{X}$, it is possible to achieve similar estimation through PGD: [OS17],[18] [S19][19]

$$\mathbf{x}^{(t)} = \mathcal{P}_{\mathcal{X}}\big(\mathbf{x}^{(t-1)} - \eta\cdot\partial\mathcal{L}_1(\mathbf{x}^{(t-1)})\big), \quad t = 1,2,\cdots \tag{19}$$

- $\mathcal{X} = \mathcal{K}\cap\mathbb{A}_{\alpha}^{\beta}$, so $\mathcal{P}_{\mathcal{X}}$ is not immediately efficient even for convex $\mathcal{K}$

  $\longrightarrow$ we use sequential projection

$$\mathcal{P}_{A_{\alpha}^{\beta}}\circ\mathcal{P}_{\mathcal{K}}$$

[18] Fast and reliable parameter estimation from nonlinear observations, S. Oymak, M. Soltanolkotabi, 2017 SIOPT

[19] Structured signal recovery from quadratic measurements: Breaking sample complexity barriers via nonconvex optimization, M. Soltanolkotabi, 2019 TIT

Projected Gradient Descent for QCS

$$\mathbf{h}(\mathbf{u},\mathbf{v}) := \frac{1}{m}\sum_{i=1}^{m}\left(Q(\mathbf{a}_i^\top \mathbf{u}-\tau_i)-Q(\mathbf{a}_i^\top \mathbf{v}-\tau_i)\right)\mathbf{a}_i \tag{20}$$

Algorithm 1 Projected Gradient Descent for QCS

1: **Input**: $\mathbf{y}=Q(\mathbf{A}\mathbf{x}-\boldsymbol{\tau})$, $(\mathbf{A},\boldsymbol{\tau})$, $\mathscr{K}$, $\mathbb{A}_\alpha^\beta$, initialization $\mathbf{x}^{(0)}$, step size η

2: **For** $t=1,2,3,\cdots$ **do**

$$\mathbf{x}^{(t)} = \mathscr{P}_{\mathbb{A}_\alpha^\beta}\left(\mathscr{P}_{\mathscr{K}}\left(\mathbf{x}^{(t-1)}-\eta\cdot\mathbf{h}(\mathbf{x}^{(t-1)},\mathbf{x})\right)\right) \tag{21}$$

- Specializing to 1bCS of $\mathbf{x}\in\Sigma_k^n\cap\mathbb{S}^{n-1}$ returns NBIHT [JLBB13], [MM24]:

$$\mathbf{x}^{(t)} = \frac{\mathsf{T}_{(k)}\left(\mathbf{x}^{(t-1)}-\eta\cdot\partial\mathscr{L}_1(\mathbf{x}^{(t-1)})\right)}{\|\mathsf{T}_{(k)}(\mathbf{x}^{(t-1)}-\eta\cdot\partial\mathscr{L}_1(\mathbf{x}^{(t-1)}))\|_2} \tag{22}$$

Restricted Approximate Invertibility Condition (RAIC)

- Dual norm: $\|\mathbf{w}\|_{\mathscr{U}^\circ} = \sup_{\mathbf{u}\in\mathscr{U}} \langle \mathbf{u}, \mathbf{w}\rangle$

Definition 1: RAIC

Under some quantizer Q, for some given $\mathscr{D} \subset \mathbb{R}^n$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4)$ with non-negative scalars $(\mu_i)_{i=1}^4$, we say $(Q, \mathbf{A}, \boldsymbol{\tau}, \mathscr{K}, \eta)$ respects $(\mathscr{D}, \boldsymbol{\mu})$-RAIC at scale $\phi > 0$ if

$$\frac{1}{\phi}\|\mathbf{u}-\mathbf{v}-\eta\cdot\mathbf{h}(\mathbf{u},\mathbf{v})\|_{\mathscr{K}^\circ_{(\phi)}} \le \mu_1\|\mathbf{u}-\mathbf{v}\|_2 + \sqrt{\mu_2\cdot\|\mathbf{u}-\mathbf{v}\|_2} + \mu_3 \quad (23)$$

holds for any $\mathbf{u}, \mathbf{v} \in \mathscr{D}$ obeying $\|\mathbf{u}-\mathbf{v}\|_2 \le \mu_4$.

Similar notions in [FJPY21],[20] [MM24]; we provide **generalization**

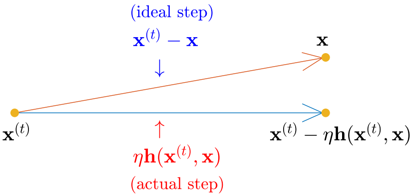
[20] NBIHT: An efficient algorithm for 1-bit compressed sensing with optimal error decay rate, M. P. Friedlander, H. Jeong, Y. Plan, Ö. Ylmaz, TIT, 2021.

Meaning of RAIC

- Meaning: the ideal descent step $\mathbf{x}^{(t)} - \mathbf{x}$ — in light of

$$\mathbf{x}^{(t)} - (\mathbf{x}^{(t)} - \mathbf{x}) = \mathbf{x}$$

— is close to the actual step $\eta \mathbf{h}(\mathbf{x}^{(t)}, \mathbf{x})$, in the dual norm sense and up to a few error terms



- Regularity Condition [CLS15]:[21] $\langle \nabla \mathscr{L}(\mathbf{u}), \mathbf{u} - \mathbf{x} \rangle \geq c_1 \|\mathbf{u} - \mathbf{x}\|_2^2 + c_2 \|\nabla \mathscr{L}(\mathbf{u})\|_2^2$

[21] Phase Retrieval via Wirtinger Flow: Theory and Algorithms, E. Candès, X. Li, M. Soltanolkotabi, TIT, 2015.

Why RAIC is useful?

Why RAIC?

$$\|\mathbf{x}^{(t)}-\mathbf{x}\|_2 = \left\|\mathscr{P}_{\mathbb{A}_\alpha^\beta}\left(\mathscr{P}_{\mathcal{K}}\left(\mathbf{x}^{(t-1)}-\eta\mathbf{h}(\mathbf{x}^{(t-1)},\mathbf{x})\right)\right)-\mathbf{x}\right\|_2$$

$$\le 2\left\|\mathscr{P}_{\mathcal{K}}\left(\mathbf{x}^{(t-1)}-\eta\mathbf{h}(\mathbf{x}^{(t-1)},\mathbf{x})\right)-\mathbf{x}\right\|_2 \qquad \blacktriangleright \text{ Definition of projection}$$

$$\le 2\max\left\{\phi, \frac{2}{\phi}\|\mathbf{x}^{(t-1)}-\mathbf{x}-\eta\mathbf{h}(\mathbf{x}^{(t-1)},\mathbf{x})\|_{\mathcal{K}^\circ_{(\phi)}}\right\} \qquad (\forall\phi>0) \tag{24}$$

► Property of projection onto star-shaped set [Corollary 8.3, PVY17]

$$\le 2\max\left\{\phi, 2\mu_1\|\mathbf{x}^{(t-1)}-\mathbf{x}\|_2+2\sqrt{\mu_2\|\mathbf{x}^{(t-1)}-\mathbf{x}\|_2}+2\mu_3\right\} \qquad \blacktriangleright \text{ RAIC} \tag{25}$$

$$\le 4\mu_1\|\mathbf{x}^{(t-1)}-\mathbf{x}\|_2+4\sqrt{\mu_2\|\mathbf{x}^{(t-1)}-\mathbf{x}\|_2}+4\mu_3 \qquad \blacktriangleright \text{ if } \phi\le\mu_3$$

Trade-off in choosing ϕ: we use $\phi=\tilde{\Theta}(\mu_3)$

RAIC - Parameters μ_i

$$\|\mathbf{x}^{(t)} - \mathbf{x}\|_2 \leq 4\mu_1 \|\mathbf{x}^{(t-1)} - \mathbf{x}\|_2 + 4\sqrt{\mu_2 \|\mathbf{x}^{(t-1)} - \mathbf{x}\|_2} + 4\mu_3 \tag{26}$$

- μ_2 and μ_3: **convergence error of PGD**
- (26) $\longrightarrow$ convergence to an ℓ_2-error of $O(\mu_2 + \mu_3)$

- μ_1: **convergence rate of PGD**
- $0 \ll \mu_1 \leq \frac{1}{4}$: linear convergence, $O(\log(\epsilon^{-1}))$ steps
- $\mu_1 = 0$: quadratic convergence, $O(\log\log(\epsilon^{-1}))$ steps

RAIC - Parameters μ_i

- μ_4: **initialization required by PGD**
- Eq. (24) $\rightarrow$ Eq. (25) requires $\|\mathbf{x}^{(t-1)} - \mathbf{x}\|_2 \le \mu_4$, so we need $\|\mathbf{x}^{(0)} - \mathbf{x}\|_2 \le \mu_4$
- μ_4 = diameter of $\mathscr{X} \longrightarrow$ global RAIC, starts with arbitrary point in the signal set
- $\mu_4 \ll$ diameter of $\mathscr{X} \longrightarrow$ local RAIC, starts with a good initialization $\|\mathbf{x}^{(0)} - \mathbf{x}\|_2 \le \mu_4$

What Remains?

Main Theorem (Informal):

If

$$m \geq C_2(\Delta \vee \Lambda)\left(\frac{\omega^2(\mathcal{K}_{(r)})}{r^3} + \frac{\log \mathcal{N}(\mathcal{X}, r/2)}{r}\right), \tag{27}$$

then $\|\hat{\mathbf{x}}_{\text{pgd}} - \mathbf{x}\|_2 = \tilde{O}(r)$

What Remains?

Prove RAIC:

$$\frac{1}{r}\left\|\mathbf{u} - \mathbf{v} - \eta \cdot \mathbf{h}(\mathbf{u},\mathbf{v})\right\|_{\mathcal{K}^\circ_{(r)}} \leq \epsilon\|\mathbf{u}-\mathbf{v}\|_2 + \sqrt{\tilde{O}(r)\|\mathbf{u}-\mathbf{v}\|_2} + \tilde{O}(r) \tag{28}$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{X}$ satisfying $\|\mathbf{u}-\mathbf{v}\|_2 \leq \mu_4$, under the sample complexity (27)

Gradient

Recall:

$$\mathbf{h}(\mathbf{u},\mathbf{v}) := \frac{1}{m}\sum_{i=1}^{m}\Big(Q(\mathbf{a}_i^\top\mathbf{u}-\tau_i)-Q(\mathbf{a}_i^\top\mathbf{v}-\tau_i)\Big)\mathbf{a}_i$$

$$= \frac{1}{m}\sum_{i=1}^{m}\mathrm{sign}(\mathbf{a}_i^\top(\mathbf{u}-\mathbf{v}))\Big|Q(\mathbf{a}_i^\top\mathbf{u}-\tau_i)-Q(\mathbf{a}_i^\top\mathbf{v}-\tau_i)\Big|\mathbf{a}_i$$

- Q is increasing

$$= \frac{1}{m}\sum_{i\in\mathbf{R}_{\mathbf{u},\mathbf{v}}}\mathrm{sign}(\mathbf{a}_i^\top(\mathbf{u}-\mathbf{v}))\Big|Q(\mathbf{a}_i^\top\mathbf{u}-\tau_i)-Q(\mathbf{a}_i^\top\mathbf{v}-\tau_i)\Big|\mathbf{a}_i$$

- $\mathbf{R}_{\mathbf{u},\mathbf{v}} := \{i\in[m] : Q(\mathbf{a}_i^\top\mathbf{u}-\tau_i)\neq Q(\mathbf{a}_i^\top\mathbf{v}-\tau_i)\}$

$\mathbf{h}(\mathbf{u},\mathbf{v})$ only contains $|\mathbf{R}_{\mathbf{u},\mathbf{v}}|$ summands!

Gradient Clipping

$$\mathbf{h}(\mathbf{u},\mathbf{v}) = \frac{1}{m}\sum_{i\in\mathbf{R}_{\mathbf{u},\mathbf{v}}} \mathrm{sign}(\mathbf{a}_i^\top(\mathbf{u}-\mathbf{v}))\left|Q(\mathbf{a}_i^\top\mathbf{u}-\tau_i)-Q(\mathbf{a}_i^\top\mathbf{v}-\tau_i)\right|\mathbf{a}_i$$

- One-bit case: if $i\in\mathbf{R}_{\mathbf{u},\mathbf{v}}$, then $|Q(\mathbf{a}_i^\top\mathbf{u}-\tau_i)-Q(\mathbf{a}_i^\top\mathbf{v}-\tau_i)|=\Delta$ and hence

$$\mathbf{h}(\mathbf{u},\mathbf{v}) = \frac{\Delta}{m}\sum_{i\in\mathbf{R}_{\mathbf{u},\mathbf{v}}} \mathrm{sign}(\mathbf{a}_i^\top(\mathbf{u}-\mathbf{v}))\mathbf{a}_i. \tag{29}$$

- Multi-bit case: if $i\in\mathbf{R}_{\mathbf{u},\mathbf{v}}$, all the following are possible

$$|Q(\mathbf{a}_i^\top\mathbf{u}-\tau_i)-Q(\mathbf{a}_i^\top\mathbf{v}-\tau_i)| = \underbrace{\Delta,\ 2\Delta,\ \cdots,\ \ell\Delta}_{\substack{\text{add to difficulty}\\ \text{forbid unified analysis}}}$$

Gradient Clipping

- We propose to clip $\mathbf{h}(\mathbf{u},\mathbf{v})$ to

$$\hat{\mathbf{h}}(\mathbf{u},\mathbf{v}) := \frac{\Delta}{m}\sum_{i\in \mathbf{R}_{\mathbf{u},\mathbf{v}}} \operatorname{sign}(\mathbf{a}_i^\top(\mathbf{u}-\mathbf{v}))\mathbf{a}_i \tag{30}$$

and seek to prove

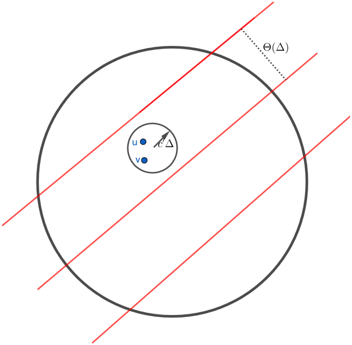
$$\frac{1}{r}\left\|\mathbf{u}-\mathbf{v}-\eta\cdot\hat{\mathbf{h}}(\mathbf{u},\mathbf{v})\right\|_{\mathcal{K}^\circ_{(r)}} \le \epsilon\|\mathbf{u}-\mathbf{v}\|_2 + \sqrt{\tilde{O}(r)\|\mathbf{u}-\mathbf{v}\|_2} + \tilde{O}(r), \forall \mathbf{u},\mathbf{v}\in\mathcal{X},\ \ \|\mathbf{u}-\mathbf{v}\|_2 \le \mu_4. \tag{31}$$

- **One-bit case**: No issue
- **Multi-bit case**: deviation $\longrightarrow \frac{\eta}{r}\|\mathbf{h}(\mathbf{u},\mathbf{v})-\hat{\mathbf{h}}(\mathbf{u},\mathbf{v})\|_{\mathcal{K}^\circ_{(r)}}$ where

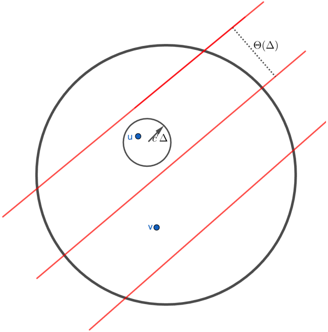
$$\mathbf{h}(\mathbf{u},\mathbf{v})-\hat{\mathbf{h}}(\mathbf{u},\mathbf{v}) = \frac{1}{m}\sum_{i\in\mathbf{R}_{\mathbf{u},\mathbf{v}}}\operatorname{sign}(\mathbf{a}_i^\top(\mathbf{u}-\mathbf{v}))\underbrace{\Big[|Q(\mathbf{a}_i^\top\mathbf{u}-\tau_i)-Q(\mathbf{a}_i^\top\mathbf{v}-\tau_i)|-\Delta\Big]}_{:=D_1}\mathbf{a}_i \tag{32}$$

How Can We Work With $\hat{\mathbf{h}}$ in Multi-Bit Case?

- **Idea: set $\mu_4 = c'\Delta$ with very small c', restrict the attention to $\|\mathbf{u}-\mathbf{v}\|_2 \le c'\Delta$**
- $\mathbb{P}(D_1 \neq 0,\ i \in \mathbf{R}_{\mathbf{u},\mathbf{v}})$ is very small!



$\mathbf{h}(\mathbf{u},\mathbf{v}) \approx \hat{\mathbf{h}}(\mathbf{u},\mathbf{v})$



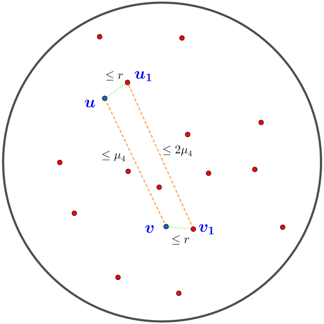
$\mathbf{h}(\mathbf{u},\mathbf{v}), \hat{\mathbf{h}}(\mathbf{u},\mathbf{v})$ differ so much

- Seek initialization $\|\mathbf{x}^{(0)} - \mathbf{x}\|_2 \le \mu_4 = c'\Delta$

Covering Argument

- Goal: bound $\frac{1}{r}\|\mathbf{u}-\mathbf{v}-\eta\cdot\hat{\mathbf{h}}(\mathbf{u},\mathbf{v})\|_{\mathcal{K}^{\circ}_{(r)}}$ for all $\mathbf{u},\mathbf{v}\in\mathcal{X}$, $\|\mathbf{u}-\mathbf{v}\|_2\le\mu_4$
- let $\mathcal{N}_r$ be a minimal r-net ($r\le\frac{\mu_4}{2}$) of $\mathcal{X}$ such that $|\mathcal{N}_r|=\mathcal{N}(\mathcal{X},r)$. Then, for any $\mathbf{u},\mathbf{v}\in\mathcal{X}$ obeying $\|\mathbf{u}-\mathbf{v}\|_2\le\mu_4$, we find their closest points in $\mathcal{N}_r$:

$$\mathbf{u}_1:=\arg\min_{\mathbf{w}\in\mathcal{N}_r}\|\mathbf{w}-\mathbf{u}\|_2 \text{ and } \mathbf{v}_1:=\arg\min_{\mathbf{w}\in\mathcal{N}_r}\|\mathbf{w}-\mathbf{v}\|_2. \quad (33)$$



What to Bound?

Passing to the net:

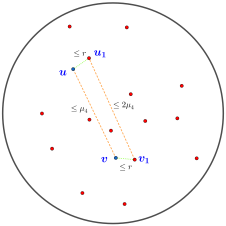
$$
\begin{aligned}
& r^{-1}\|\mathbf{u}-\mathbf{v}-\eta\cdot\hat{\mathbf{h}}(\mathbf{u},\mathbf{v})\|_{\mathcal{K}^{\circ}_{(r)}} \\
& \leq \underbrace{r^{-1}\|\mathbf{u}_1-\mathbf{v}_1-\eta\cdot\hat{\mathbf{h}}(\mathbf{u}_1,\mathbf{v}_1)\|_{\mathcal{K}^{\circ}_{(r)}}}_{\text{bound on finite net}} + \underbrace{r^{-1}\|\mathbf{u}-\mathbf{u}_1\|_{\mathcal{K}^{\circ}_{(r)}} + r^{-1}\|\mathbf{v}-\mathbf{v}_1\|_{\mathcal{K}^{\circ}_{(r)}}}_{\leq 2r} \\
& \quad + \underbrace{\frac{\eta}{r}\left(\|\hat{\mathbf{h}}(\mathbf{u}_1,\mathbf{v}_1)-\hat{\mathbf{h}}(\mathbf{u},\mathbf{v})\|_{\mathcal{K}^{\circ}_{(r)}}\right)}_{\text{approximation gap}}
\end{aligned}
$$

Approximation Gap

- Bound $\frac{\eta}{r}\left(\|\hat{\mathbf{h}}(\mathbf{u}_1,\mathbf{v}_1)-\hat{\mathbf{h}}(\mathbf{u},\mathbf{v})\|_{\mathcal{K}^{\circ}_{(r)}}\right)$ for all $(\mathbf{u},\mathbf{v})$ and corresponding $(\mathbf{u}_1,\mathbf{v}_1)$
- Recall: $\mathbf{h}(\mathbf{p},\mathbf{q})$ contains only $|\mathbf{R}_{\mathbf{p},\mathbf{q}}|$ summands.
- Reducing number of nonzero contributors:

$$\underbrace{\hat{\mathbf{h}}(\mathbf{u}_1,\mathbf{v}_1)}_{|\mathbf{R}_{\mathbf{u}_1,\mathbf{v}_1}|}-\underbrace{\hat{\mathbf{h}}(\mathbf{u},\mathbf{v})}_{|\mathbf{R}_{\mathbf{u},\mathbf{v}}|}\approx \mathbf{h}(\mathbf{u}_1,\mathbf{v}_1)-\mathbf{h}(\mathbf{u},\mathbf{v})=\mathbf{h}(\mathbf{u}_1,\mathbf{u})+\mathbf{h}(\mathbf{v},\mathbf{v}_1)\approx \underbrace{\hat{\mathbf{h}}(\mathbf{u}_1,\mathbf{u})}_{|\mathbf{R}_{\mathbf{u}_1,\mathbf{u}}|}+\underbrace{\hat{\mathbf{h}}(\mathbf{v},\mathbf{v}_1)}_{|\mathbf{R}_{\mathbf{v},\mathbf{v}_1}|}$$

- Stronger closeness: $\|\mathbf{u}-\mathbf{v}\|_2\le\mu_4,\ \|\mathbf{u}_1-\mathbf{v}_1\|_2\le 2\mu_4 \quad\longrightarrow\quad \|\mathbf{u}-\mathbf{u}_1\|_2\le r,\ \|\mathbf{v}-\mathbf{v}_1\|_2\le r$



Approximation Gap

- Very few nonzero contributors: $|\mathbf{R}_{\mathbf{u}_1,\mathbf{u}}| + |\mathbf{R}_{\mathbf{v},\mathbf{v}_1}| = \tilde{O}(\frac{mr}{\Delta\vee\Lambda})$
  - ▶ by *quantized embedding property* (small to small)

Crude argument suffices:

$$\frac{\eta}{r}\|\hat{\mathbf{h}}(\mathbf{u}_1,\mathbf{u})\|_{\mathscr{K}^{\circ}_{(r)}}$$

$$= \sup_{\mathbf{w}\in\mathscr{K}_{(r)}} \frac{\Delta\eta}{mr} \sum_{i\in\mathbf{R}_{\mathbf{u}_1,\mathbf{u}}} \operatorname{sign}(\mathbf{a}_i^\top(\mathbf{u}_1-\mathbf{u}))\mathbf{a}_i^\top\mathbf{w}$$

$$\leq \sup_{\mathbf{w}\in\mathscr{K}_{(r)}} \frac{\Delta\eta}{mr} \max_{\substack{I\subset[m]\\ |I|=\tilde{O}(\frac{mr}{\Delta\vee\Lambda})}} \sum_{i\in I}|\mathbf{a}_i^\top\mathbf{w}|$$

▶ by triangle inquality and $|\mathbf{R}_{\mathbf{u}_1,\mathbf{u}}| = \tilde{O}\left(\frac{mr}{\Delta\vee\Lambda}\right)$

$$= \frac{\Delta\eta}{\Delta\vee\Lambda}\tilde{O}(r) = \tilde{O}(r)$$

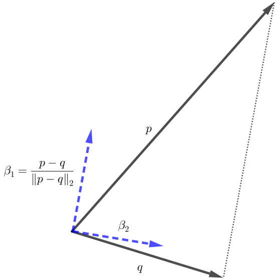
▶ by a well-known bound

Bound on Finite Net - Orthogonal Decomposition

- Bound $\frac{1}{r}\|\mathbf{p}-\mathbf{q}-\eta\cdot\hat{\mathbf{h}}(\mathbf{p},\mathbf{q})\|_{\mathcal{K}^{\circ}_{(r)}}$ for all $\mathbf{p},\mathbf{q}\in\mathcal{N}_r$ satisfying $\|\mathbf{p}-\mathbf{q}\|_2\le 2\mu_4$
- Find $\left(\boldsymbol{\beta}_1 := \frac{\mathbf{p}-\mathbf{q}}{\|\mathbf{p}-\mathbf{q}\|_2},\ \boldsymbol{\beta}_2\right)$ as orthonormal basis in span$(\mathbf{p},\mathbf{q})$ such that

$$\mathbf{p} = u_1\boldsymbol{\beta}_1 + u_2\boldsymbol{\beta}_2 \ \text{ and } \ \mathbf{q} = v_1\boldsymbol{\beta}_1 + u_2\boldsymbol{\beta}_2 \tag{34}$$

for some coordinates (u_1, u_2, v_1)



Bound on Finite Net - Orthogonal Decomposition

- Decompose $\hat{\mathbf{h}}(\mathbf{p},\mathbf{q})$ along three directions: $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, $\{\text{span}(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2)\}^\perp$

$$\hat{\mathbf{h}}(\mathbf{p},\mathbf{q}) = \langle\hat{\mathbf{h}}(\mathbf{p},\mathbf{q}),\boldsymbol{\beta}_1\rangle\boldsymbol{\beta}_1 + \langle\hat{\mathbf{h}}(\mathbf{p},\mathbf{q}),\boldsymbol{\beta}_2\rangle\boldsymbol{\beta}_2 + [\underbrace{\hat{\mathbf{h}}(\mathbf{p},\mathbf{q}) - \langle\hat{\mathbf{h}}(\mathbf{p},\mathbf{q}),\boldsymbol{\beta}_1\rangle\boldsymbol{\beta}_1 - \langle\hat{\mathbf{h}},(\mathbf{p},\mathbf{q}),\boldsymbol{\beta}_2\rangle\boldsymbol{\beta}_2}_{:=\hat{\mathbf{h}}^\perp(\mathbf{p},\mathbf{q})}]$$

- Substituting into $\frac{1}{r}\|\mathbf{p}-\mathbf{q}-\eta\hat{\mathbf{h}}(\mathbf{p},\mathbf{q})\|_{\mathcal{K}^\circ_{(r)}}$:

$$\begin{aligned}
&\frac{1}{r}\|\mathbf{p}-\mathbf{q}-\eta\cdot\hat{\mathbf{h}}(\mathbf{p},\mathbf{q})\|_{\mathcal{K}^\circ_{(r)}} \\
&\leq \big|\|\mathbf{p}-\mathbf{q}\|_2 - \underbrace{\eta\cdot\langle\hat{\mathbf{h}}(\mathbf{p},\mathbf{q}),\boldsymbol{\beta}_1\rangle}_{:=T_1^{\mathbf{p},\mathbf{q}}}\big| + \big|\underbrace{\eta\cdot\langle\hat{\mathbf{h}}(\mathbf{p},\mathbf{q}),\boldsymbol{\beta}_2\rangle}_{:=T_2^{\mathbf{p},\mathbf{q}}}\big| + r^{-1}\cdot\underbrace{\eta\|\hat{\mathbf{h}}^\perp(\mathbf{p},\mathbf{q})\|_{\mathcal{K}^\circ_{(r)}}}_{:=T_3^{\mathbf{p},\mathbf{q}}} \\
&:= \big|\|\mathbf{p}-\mathbf{q}\|_2 - T_1^{\mathbf{p},\mathbf{q}}\big| + |T_2^{\mathbf{p},\mathbf{q}}| + r^{-1}\cdot T_3^{\mathbf{p},\mathbf{q}}.
\end{aligned} \tag{35}$$

Bound $|\|\mathbf{p}-\mathbf{q}\|_2 - T_1^{\mathbf{p},\mathbf{q}}|$

$$T_1^{\mathbf{p},\mathbf{q}} := \eta \cdot \langle \hat{\mathbf{h}}(\mathbf{p},\mathbf{q}), \boldsymbol{\beta}_1 \rangle = \frac{\eta\Delta}{m} \sum_{i=1}^{m} |\mathbf{a}_i^\top \boldsymbol{\beta}_1| \mathbb{1}(E_{\mathbf{p},\mathbf{q}}^{(i)})$$

Assumption 4: Moment Bound I

For $\mathbf{p},\mathbf{q} \in \mathscr{X}$ satisfying $\|\mathbf{p}-\mathbf{q}\|_2 \le 2\mu_4$, we suppose

$$\mathbb{E}\Big(|\mathbf{a}_i^\top \boldsymbol{\beta}_1|^p \mathbb{1}(E_{\mathbf{p},\mathbf{q}}^{(i)})\Big) \le c^{(5)} \mathsf{P}_{\mathbf{p},\mathbf{q}} \cdot \frac{p!}{2}, \quad \forall p \ge 2 \tag{36a}$$

$$\mathbb{E}\big(|\mathbf{a}_i^\top \boldsymbol{\beta}_1| \mathbb{1}(E_{\mathbf{p},\mathbf{q}}^{(i)})\big) \approx \frac{c^{(6)} \|\mathbf{p}-\mathbf{q}\|_2}{\Delta \vee \Lambda}, \tag{36b}$$

hold for some $c^{(5)}, c^{(6)} > 0$

Bernstein's inequality (under $\eta = \frac{\Delta \vee \Lambda}{\Delta c^{(6)}}$):

$$|\|\mathbf{p}-\mathbf{q}\|_2 - T_1^{\mathbf{p},\mathbf{q}}| \lesssim \sqrt{\frac{[\Delta \vee \Lambda]\log(\mathscr{N}(\mathscr{X},r))\|\mathbf{p}-\mathbf{q}\|_2}{m}} + \frac{[\Delta \vee \Lambda]\log(\mathscr{N}(\mathscr{X},r))}{m}$$

$$\overset{(27)}{\le} \sqrt{O(r)\|\mathbf{p}-\mathbf{q}\|_2} + O(r)$$

Bound $|T_2^{\mathbf{p},\mathbf{q}}|$

$$T_2^{\mathbf{p},\mathbf{q}} := \eta \cdot \langle \hat{\mathbf{h}}(\mathbf{p},\mathbf{q}), \boldsymbol{\beta}_2 \rangle = \frac{\eta\Delta}{m} \sum_{i=1}^{m} \mathrm{sign}(\mathbf{a}_i^\top \boldsymbol{\beta}_1)(\mathbf{a}_i^\top \boldsymbol{\beta}_2) \mathbb{1}(E_{\mathbf{p},\mathbf{q}}^{(i)}),$$

Assumption 5: Moment Bound II

For any $\mathbf{p},\mathbf{q} \in \mathscr{X}$ obeying $0 < \|\mathbf{p}-\mathbf{q}\|_2 \leq 2\mu_4$, we suppose

$$\mathbb{E}\Big(|\mathbf{a}_i^\top \boldsymbol{\beta}_2|^p \mathbb{1}(E_{\mathbf{p},\mathbf{q}}^{(i)})\Big) \leq c^{(7)} \mathsf{P}_{\mathbf{p},\mathbf{q}} \cdot \frac{p!}{2}, \quad \forall p \geq 2 \tag{37a}$$

$$\mathbb{E}\Big(\mathrm{sign}(\mathbf{a}_i^\top \boldsymbol{\beta}_1)\mathbf{a}_i^\top \boldsymbol{\beta}_2 \mathbb{1}(E_{\mathbf{p},\mathbf{q}}^{(i)})\Big) \approx 0, \tag{37b}$$

hold for some $c^{(7)} > 0$

Bernstein's inequality:

$$|T_2^{\mathbf{p},\mathbf{q}}| \lesssim \sqrt{\frac{[\Delta \vee \Lambda] \log(\mathscr{N}(\mathscr{X}, r)) \|\mathbf{p}-\mathbf{q}\|_2}{m}} + \frac{[\Delta \vee \Lambda] \log(\mathscr{N}(\mathscr{X}, r))}{m}$$

$$\overset{(27)}{\leq} \sqrt{O(r)\|\mathbf{p}-\mathbf{q}\|_2} + O(r)$$

Bound $r^{-1} T_3^{\mathbf{p},\mathbf{q}}$

$$T_3^{\mathbf{p},\mathbf{q}} := \sup_{\mathbf{w}\in\mathcal{K}_{(r)}} \eta \cdot \mathbf{w}^\top \hat{\mathbf{h}}^\perp(\mathbf{p},\mathbf{q}) = \sup_{\mathbf{w}\in\mathcal{K}_{(r)}} \frac{\eta\Delta}{m} \sum_{i=1}^{m} \underbrace{\mathrm{sign}(\mathbf{a}_i^\top \boldsymbol{\beta}_1)\big[\mathbf{a}_i - (\mathbf{a}_i^\top \boldsymbol{\beta}_1)\boldsymbol{\beta}_1 - (\mathbf{a}_i^\top \boldsymbol{\beta}_2)\boldsymbol{\beta}_2\big]^\top \mathbf{w}}_{:=J_{i,\mathbf{w}}^{\mathbf{p},\mathbf{q}}} \cdot \mathbb{1}(E_{\mathbf{p},\mathbf{q}}^{(i)}).$$

Assumption 6: Moment Bound III

For any $\mathbf{w} \in \mathbb{R}^n$ and any $\mathbf{p},\mathbf{q} \in \mathscr{X}$ obeying $0 < \|\mathbf{p}-\mathbf{q}\|_2 \le 2\mu_4$, we suppose

$$\mathbb{E}\Big(|J_{i,\mathbf{w}}^{\mathbf{p},\mathbf{q}}|^p \mathbb{1}(E_{\mathbf{p},\mathbf{q}}^{(i)})\Big) \le \mathsf{P}_{\mathbf{p},\mathbf{q}}(c^{(8)}\sqrt{p})^p, \quad \forall p \ge 2, \ \forall \mathbf{w} \in \mathbb{S}^{n-1} \tag{38a}$$

$$\mathbb{E}\Big(J_{i,\mathbf{w}}^{\mathbf{p},\mathbf{q}} \mathbb{1}(E_{\mathbf{p},\mathbf{q}}^{(i)})\Big) \approx 0, \ \forall \mathbf{w} \in \mathbb{S}^{n-1} \tag{38b}$$

hold for some $c^{(8)} > 0$, $\varepsilon^{(4)} \ge 0$.

Bound $r^{-1} T_3^{\mathbf{p},\mathbf{q}}$

$$T_3^{\mathbf{p},\mathbf{q}} = \sup_{\mathbf{w}\in\mathcal{K}_{(r)}} \frac{\eta\Delta}{m} \sum_{i\in\mathbf{R}_{\mathbf{p},\mathbf{q}}} J_{i,\mathbf{w}}^{\mathbf{p},\mathbf{q}}$$

Conditional Concentration: (similar to [MM24])

- We condition on $|\mathbf{R}_{\mathbf{p},\mathbf{q}}| = r_{\mathbf{p},\mathbf{q}}$, then

$$T_3^{\mathbf{p},\mathbf{q}} \sim \sup_{\mathbf{w}\in\mathcal{K}_{(r)}} \frac{\eta\Delta}{m} \sum_{i=1}^{r_{\mathbf{p},\mathbf{q}}} \hat{J}_i(\mathbf{w}), \quad \hat{J}_i(\mathbf{w}) \overset{iid}{\sim} J_{i,\mathbf{w}}^{\mathbf{p},\mathbf{q}} | \{E_{\mathbf{p},\mathbf{q}}^{(i)}\}$$

- Moment bound in Assumption 7 states $\|\hat{J}_i(\mathbf{w})\|_{\psi_2} = O(c^{(8)})$, so we use Talagrand's and capture $\omega(\mathcal{K}_{(r)})$

Chernoff Inequality:

- It is easy to get rid of the conditioning: $r_{\mathbf{p},\mathbf{q}} \sim \mathrm{Bin}(m, \mathsf{P}_{\mathbf{p},\mathbf{q}})$

Final bound:

$$r^{-1} T_3^{\mathbf{p},\mathbf{q}} \lesssim \sqrt{\frac{\|\mathbf{p}-\mathbf{q}\|_2 [\Delta \vee \Lambda][\frac{\omega^2(\mathcal{K}_{(r)})}{r^2} + \log \mathcal{N}(\mathcal{X}, r)]}{m}} + \frac{[\Delta \vee \Lambda][\frac{\omega^2(\mathcal{K}_{(r)})}{r^2} + \log \mathcal{N}(\mathcal{X}, r)]}{m}$$
$$\overset{(27)}{\leq} \sqrt{O(r)\|\mathbf{p}-\mathbf{q}\|_2} + O(r)$$

Proof is Complete!

Combining all the pieces:

$$\frac{\eta}{r}\left(\|\hat{\mathbf{h}}(\mathbf{u}_1,\mathbf{u})\|_{\mathcal{K}^\circ_{(r)}} + \|\hat{\mathbf{h}}(\mathbf{v},\mathbf{v}_1)\|_{\mathcal{K}^\circ_{(r)}}\right) = \tilde{O}(r)$$

$$|\|\mathbf{p}-\mathbf{q}\|_2 - T_1^{\mathbf{p},\mathbf{q}}| \le \sqrt{O(r)\|\mathbf{p}-\mathbf{q}\|_2} + O(r)$$

$$|T_2^{\mathbf{p},\mathbf{q}}| \le \sqrt{O(r)\|\mathbf{p}-\mathbf{q}\|_2} + O(r)$$

$$r^{-1}\,T_3^{\mathbf{p},\mathbf{q}} \le \sqrt{O(r)\|\mathbf{p}-\mathbf{q}\|_2} + O(r)$$

the desired RAIC follows!

Understanding the Moment Bound (Vague)

Moment bounds in Assumption 5:

$$\mathbb{E}\Big(|\mathbf{a}_i^\top \boldsymbol{\beta}_1|^p \mathbb{1}(E_{\mathbf{p},\mathbf{q}}^{(i)})\Big) \le c^{(5)} \mathsf{P}_{\mathbf{p},\mathbf{q}} \cdot \frac{p!}{2}, \quad \forall p \ge 2 \tag{39}$$

Kind of "*independence*" between $\mathbf{a}_i^\top \boldsymbol{\beta}_1$ and $E_{\mathbf{p},\mathbf{q}}^{(i)}$:

- Cauchy-Schwarz gives

$$\mathbb{E}\Big(|\mathbf{a}_i^\top \boldsymbol{\beta}_1|^p \mathbb{1}(E_{\mathbf{p},\mathbf{q}}^{(i)})\Big) \le \sqrt{\mathbb{E}|\mathbf{a}_i^\top \boldsymbol{\beta}_1|^{2p}}\sqrt{\mathbb{P}(E_{\mathbf{p},\mathbf{q}}^{(i)})} \le \frac{c^{(5)} p!}{2}\sqrt{\mathsf{P}_{\mathbf{p},\mathbf{q}}} \tag{40}$$

  but the bound is not strong enough when $\mathsf{P}_{\mathbf{p},\mathbf{q}} = o(1)$
  (Cauchy-Schwarz is tight if $|\mathbf{a}_i^\top \boldsymbol{\beta}_1|^p = t\mathbb{1}(E_{\mathbf{p},\mathbf{q}}^{(i)})$ for some $t > 0$)

- When $\mathbf{a}_i^\top \boldsymbol{\beta}_1$ and $E_{\mathbf{p},\mathbf{q}}^{(i)}$ are independent:

$$\mathbb{E}\Big(|\mathbf{a}_i^\top \boldsymbol{\beta}_1|^p \mathbb{1}(E_{\mathbf{p},\mathbf{q}}^{(i)})\Big) = \mathbb{E}\big(|\mathbf{a}_i^\top \boldsymbol{\beta}_1|^p\big)\mathbb{E}\big(\mathbb{1}(E_{\mathbf{p},\mathbf{q}}^{(i)})\big) \le \frac{c^{(5)} p!}{2}\mathsf{P}_{\mathbf{p},\mathbf{q}} \tag{41}$$

  we are done!

Why Moment Bounds are Satisfied by (D)1bCS & DMbCS?

Moment bound in Assumption 5:

$$\mathbb{E}\Big(|\mathbf{a}_i^\top \boldsymbol{\beta}_1|^p \mathbb{1}(E_{\mathbf{p},\mathbf{q}}^{(i)})\Big) \le c^{(5)} \mathsf{P}_{\mathbf{p},\mathbf{q}} \cdot \frac{p!}{2}, \quad \forall p \ge 2$$

where

$$\begin{aligned} E_{\mathbf{p},\mathbf{q}}^{(i)} &= \Big\{ Q(\mathbf{a}_i^\top \mathbf{p} - \tau_i) \neq Q(\mathbf{a}_i^\top \mathbf{q} - \tau_i) \Big\} \\ &= \Big\{ Q(u_1 \mathbf{a}_i^\top \boldsymbol{\beta}_1 + u_2 \mathbf{a}_i^\top \boldsymbol{\beta}_2 - \tau_i) \neq Q(v_1 \mathbf{a}_i^\top \boldsymbol{\beta}_1 + u_2 \mathbf{a}_i^\top \boldsymbol{\beta}_2 - \tau_i) \Big\} \end{aligned} \tag{42}$$

1bCS: $\mathbf{a}_i^\top \boldsymbol{\beta}_1, \mathbf{a}_i^\top \boldsymbol{\beta}_2 \overset{iid}{\sim} \mathcal{N}(0,1)$

$$\mathbb{E}\Big(|\mathbf{a}_i^\top \boldsymbol{\beta}_1|^p \mathbb{1}(E_{\mathbf{p},\mathbf{q}}^{(i)})\Big) = \mathbb{E}\Big(|\mathbf{a}_i^\top \boldsymbol{\beta}_1|^p \cdot \mathbb{P}_{\mathbf{a}_i^\top \boldsymbol{\beta}_2 \sim \mathcal{N}(0,1)}[E_{\mathbf{p},\mathbf{q}}^{(i)}]\Big) \tag{43}$$

D1bCS & DMbCS:

$$\mathbb{E}\Big(|\mathbf{a}_i^\top \boldsymbol{\beta}_1|^p \mathbb{1}(E_{\mathbf{p},\mathbf{q}}^{(i)})\Big) = \mathbb{E}\Big(|\mathbf{a}_i^\top \boldsymbol{\beta}_1|^p \cdot \mathbb{P}_{\tau_i \sim \mathscr{U}[-\Lambda,\Lambda]}[E_{\mathbf{p},\mathbf{q}}^{(i)}]\Big)$$

Concluding Remarks

This Talk:

- RAIC is useful in other nonconvex optimization problems
- Under sub-Gaussian **A**, the HMD rates only require two essential components — small-ball probability and separation probability
- Under some additional moment bounds, PGD achieves the same rate as HDM
- Validating these assumptions, the PGD rates improve on or match the best known rates in all instances. [CY24b][22]

Questions:

- The tightness of $O((\frac{k}{m})^{1/3})$ in recovering signals in $\sqrt{k}\mathbb{B}_1^n$;
- Optimal algorithms in structured sensing matrices;
- Optimal QCS of signals in a generative prior (may not be star-shaped)?
- ...

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Thank you for listening

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