Phase Transitions in Phase-Only Compressed Sensing

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Outline

Problem Setup

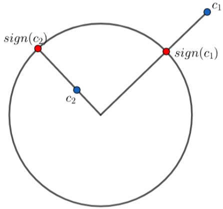
- Goal: Recover s-sparse signal $\mathbf{x} \in \mathbb{S}^{n-1}$ from $\mathbf{z} = \text{sign}(\mathbf{\Phi x})$, $\mathbf{\Phi} \in \mathbb{C}^{m \times n}$, i.e.,

$$z_i = \text{sign}(\mathbf{\Phi}_i^* \mathbf{x}), \qquad i \in [m] \tag{1}$$

- For $c \in \mathbb{C} \setminus \{0\}$ we have

$$c = |c| \cdot \frac{c}{|c|} = \text{magnitude} \cdot \text{phase} \tag{2}$$

- Let $\text{sign}(c) = \frac{c}{|c|}$



Motivations

- Applications in quantization & robustness.
- A complex version of one-bit compressed sensing
  - Recover s-sparse $\mathbf{x} \in \mathbb{S}^{n-1}$ from $\mathbf{y} = \mathrm{sign}(\mathbf{A}\mathbf{x})$, $\mathbf{A} \in \mathbb{R}^{m\times n}$, i.e.,

$$y_i = \mathrm{sign}(\mathbf{a}_i^\top \mathbf{x}), \qquad i \in [m] \tag{3}$$

  - Optimal error rate $\|\hat{\mathbf{x}} - \mathbf{x}\|_2 = \tilde{\Theta}(\frac{s}{m})$ [JLBB13][1]
- The opposite of phase retrieval
  - Recover $\mathbf{x}$ from $\mathbf{y} = |\mathbf{\Phi}\mathbf{x}|$

[1] L. Jacques et al, Robust 1-bit compressive sensing via binary stable embeddings of sparse vectors. TIT, 2013.

Prior Work

- Before 2015: Studied by Boufounos; only showed approximate recovery but numerically observed exact recovery [B13]
- Recently revisited by Jacques and coauthors:
  - **A linearization approach achieves exact reconstruction of sparse signals** [JF21][2]

[2] **The importance of phase in complex compressive sensing. L. Jacques & T. Feuillen, TIT, 2021.**

Linearization Approach [JF21]

Linearization:

- $\mathbf{z} = \operatorname{sign}(\mathbf{\Phi x})$ implies

$$\frac{1}{\sqrt{m}} \Im(\operatorname{diag}(\mathbf{z}^*)\mathbf{\Phi})\mathbf{u} = 0 \tag{4}$$

- To specify signal norm we also add

$$\frac{1}{m} \Re(\mathbf{z}^* \mathbf{\Phi})\mathbf{u} = 1. \tag{5}$$

- (5) and (4) can be concisely written as

$$\mathbf{A_z u} = \mathbf{e}_1, \qquad \text{where } \mathbf{A_z} := \begin{bmatrix} \frac{1}{m}\Re(\mathbf{z}^* \mathbf{\Phi}) \\ \frac{1}{\sqrt{m}}\Im(\operatorname{diag}(\mathbf{z}^*)\mathbf{\Phi}) \end{bmatrix} \in \mathbb{R}^{(m+1)\times n} \tag{6}$$

Algorithm: Compute $\mathbf{x}^\sharp = \frac{\hat{\mathbf{x}}}{\|\hat{\mathbf{x}}\|_2}$ where

$$\hat{\mathbf{x}} = \arg\min\ f(\mathbf{u}) = \|\mathbf{u}\|_1, \qquad \text{subject to } \mathbf{A_z u} = \mathbf{e}_1 \tag{7}$$

$\mathbf{A_z}$ Satisfies RIP

- $\mathbf{\Phi}$ has iid $\mathcal{N}(0,1)+\mathcal{N}(0,1)\mathbf{i}$ entries
- Fix $\mathbf{x}\in\mathbb{S}^{n-1}$. For cone $\mathcal{K}$, we have $\mathbf{A_z}\sim \mathrm{RIP}(\mathcal{K},\delta)$, i.e.,

$$||\|\mathbf{A_z}\mathbf{u}\|_2^2-\|\mathbf{u}\|_2^2|\le\delta\|\mathbf{u}\|_2^2,\qquad \forall\mathbf{u}\in\mathcal{K}. \tag{8}$$

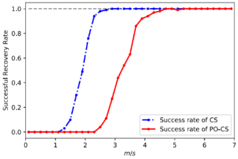
w.h.p. if $m\ge C\delta^{-2}\omega^2\Big((\mathcal{K}-\mathbb{R}\mathbf{x})\cap\mathbb{S}^{n-1}\Big)$ [JF21].

- Sparse recovery: When $m=O(s\log\frac{en}{s})$, then $\mathbf{A_z}\sim\mathrm{RIP}(\Sigma^n_{2s},\frac{1}{3})$, and hence under $f(\mathbf{u})=\|\mathbf{u}\|_1$, $\mathbf{x}^\sharp=\mathbf{x}$ w.h.p.

Question 1: Phase Transition

What is the precise number of measurements $\zeta_{\mathrm{PO}}(\mathbf{x};f)$ needed for achieving $\mathbf{x}^\sharp=\mathbf{x}$?

A Conjecture in [JF21]



- We let $\zeta_{\mathrm{LN}}(\mathbf{x};f)$ be the phase transition location of recovering $\mathbf{x}$ from $\mathbf{y}=\mathbf{A}\mathbf{x}$ $(\mathbf{A}\sim\mathcal{N}^{m\times n}(0,1))$ via

$$\min\ f(\mathbf{u})(=\|\mathbf{u}\|_1),\quad \text{subject to } \mathbf{A}\mathbf{u}=\mathbf{y}.$$

- $\zeta_{\mathrm{LN}}(\mathbf{x};f)=\delta\big(T_f(\mathbf{x})\big)$ [ALMT14][3]
    - $\delta(\mathcal{K})$ is the statistical dimension of cone $\mathcal{K}$;
    - $T_f(\mathbf{x})=\{\mathbf{u}\in\mathbb{R}^n:\exists t>0 \text{ s.t. } f(\mathbf{x}+t\mathbf{u})\le f(\mathbf{x})\}$: descent cone of f at $\mathbf{x}$
- A Conjecture [JF21]: $\zeta_{\mathrm{PO}}(\mathbf{x};f)\approx\zeta_{\mathrm{LN}}(\mathbf{x};f)$

Question 2: The Conjecture

Can we rigorously prove or disprove $\zeta_{\mathrm{PO}}\approx\zeta_{\mathrm{LN}}$?

[3] **Living on the edge: Phase transitions in convex programs with random data. D. Amelunxen, M. Lotz, M. McCoy, J. Tropp, Inf. & Inference, 2014.**

Our Contributions

- We show

$$\zeta_{\mathrm{PO}}(\mathbf{x};f) = \left[\mathbb{E} \sup_{\substack{\mathbf{u}\in T_f(\mathbf{x}) \\ \|\mathbf{Q}_{\mathbf{x}}\mathbf{u}\|_2=1}} \langle (\mathbf{I}_n - \mathbf{x}\mathbf{x}^\top)\mathbf{g}, \mathbf{u} \rangle \right]^2 \tag{9}$$

  - $T_f(\mathbf{x}) = \{\mathbf{u}\in\mathbb{R}^n : \exists t>0 \text{ s.t. } f(\mathbf{x}+t\mathbf{u}) \le f(\mathbf{x})\}$: descent cone of f at $\mathbf{x}$
  - $\mathbf{Q}_{\mathbf{x}} = \mathbf{I}_n + (\sqrt{\frac{\pi}{2}} - 1)\mathbf{x}\mathbf{x}^\top$, hence $\mathbf{Q}_{\mathbf{x}}^{-1} = \mathbf{I}_n - (1-\sqrt{2/\pi})\mathbf{x}\mathbf{x}^\top$

- We derive explicit formula for $\zeta_{\mathrm{PO}}(\mathbf{x};\|\cdot\|_1)$

Our Contributions

- We disprove the conjecture and show that $\frac{\zeta_{\mathrm{PO}}}{\zeta_{\mathrm{LN}}} \le 1$ is bounded away from 1 in the sparse case
- E.g., $\zeta_{\mathrm{PO}}(\mathbf{x};\ell_1) < 0.75 \cdot \zeta_{\mathrm{LN}}(\mathbf{x};\ell_1)$ for 1-sparse $\mathbf{x} \in \mathbb{S}^{999}$.

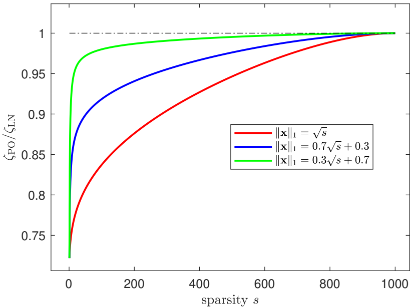


Figure 1: We fix $n = 1000$ and plot the (approximate) curves of $\zeta_{\mathrm{PO}}/\zeta_{\mathrm{LN}}$ v.s. $s = 1:1000$ under $\|\mathbf{x}\|_1 = \sqrt{s},\ 0.7\sqrt{s}+0.3,\ 0.3\sqrt{s}+0.7$.

Main Theorem

Theorem 1: Phase Transition Threshold

Suppose that the entries of $\mathbf{\Phi}$ are i.i.d. $\mathcal{N}(0,1)+\mathcal{N}(0,1)\mathbf{i}$, and consider the recovery of a fixed signal $\mathbf{x}\in\mathbb{S}^{n-1}$ from $\mathbf{z}=\operatorname{sign}(\mathbf{\Phi x})$. Define

$$\zeta_{\mathrm{PO}}(\mathbf{x};f)=\left[\mathbb{E}\sup_{\substack{\mathbf{u}\in T_f(\mathbf{x})\\ \|\mathbf{Q}_{\mathbf{x}}\mathbf{u}\|_2=1}}\langle(\mathbf{I}_n-\mathbf{x}\mathbf{x}^\top)\mathbf{g},\mathbf{u}\rangle\right]^2 \tag{10}$$

There exists an absolute constant c, such that for any $t\in[\frac{17}{m},c]$, we have:

- If $m\geq(1+t)\cdot\zeta_{\mathrm{PO}}(\mathbf{x};f)$, then

$$\mathbb{P}(\mathbf{x}^\sharp=\mathbf{x})\geq 1-14\exp\left(-\frac{mt^2}{289}\right);$$

- If $m\leq(1-t)\cdot\zeta_{\mathrm{PO}}(\mathbf{x};f)$, then

$$\mathbb{P}(\mathbf{x}^\sharp\neq\mathbf{x})\geq 1-14\exp\left(-\frac{mt^2}{289}\right),$$

where $\zeta_{\mathrm{PO}}(\mathbf{x};f)$ is the quantity defined in (9).

Conditions for Success and Failure

We first identify the conditions for success ($\mathbf{x}^\sharp = \mathbf{x}$) and failure ($\mathbf{x}^\sharp \neq \mathbf{x}$).

Lemma 1: Conditions for success and failure

For a fixed $\mathbf{x} \in \mathbb{S}^{n-1}$, suppose that $\|\mathbf{\Phi x}\|_1 > 0$. Let $f(\cdot)$ denote a norm in $\mathbb{R}^n$, and assume that $T_f(\mathbf{x})$ is closed and not a subspace. Then the following two statements are correct:

- We have $\mathbf{x}^\sharp = \mathbf{x}$ if

$$\min_{\mathbf{u} \in T_f^*(\mathbf{x})} \max_{\mathbf{v} \in \mathbb{S}^m} \mathbf{v}^\top \mathbf{A_z u} > 0. \tag{11}$$

- We have $\mathbf{x}^\sharp \neq \mathbf{x}$ if

$$\min_{\mathbf{v} \in \mathbb{S}^m} \max_{\mathbf{u} \in T_f^*(\mathbf{x})} \mathbf{v}^\top \mathbf{A_z u} > 0. \tag{12}$$

Gaussian Min-Max Theorem

Lemma 2: Gaussian min-max theorem

We let $\mathbf{G} \in \mathbb{R}^{m\times(n-1)}$, $\mathbf{g} \in \mathbb{R}^m$, $\mathbf{h} \in \mathbb{R}^{n-1}$, $\mathcal{S}_{\mathbf{w}} \subset \mathbb{R}^n$, $\mathcal{S}_{\mathbf{u}} \subset \mathbb{R}^m$, $\psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $\mathbf{w} = (w_1, \tilde{\mathbf{w}}^\top)^\top \in \mathcal{S}_{\mathbf{w}}$ with $\tilde{\mathbf{w}} \in \mathbb{R}^{n-1}$, and define

$$\Gamma(\mathbf{G}) := \min_{\mathbf{w}\in\mathcal{S}_{\mathbf{w}}} \max_{\mathbf{u}\in\mathcal{S}_{\mathbf{u}}} \mathbf{u}^\top \mathbf{G}\tilde{\mathbf{w}} + \psi(\mathbf{w},\mathbf{u}),$$

$$\Upsilon(\mathbf{g},\mathbf{h}) := \min_{\mathbf{w}\in\mathcal{S}_{\mathbf{w}}} \max_{\mathbf{u}\in\mathcal{S}_{\mathbf{u}}} \|\tilde{\mathbf{w}}\|_2 \mathbf{g}^\top \mathbf{u} + \|\mathbf{u}\|_2 \mathbf{h}^\top \tilde{\mathbf{w}} + \psi(\mathbf{w},\mathbf{u}).$$

Assume that $\mathcal{S}_{\mathbf{w}}$ and $\mathcal{S}_{\mathbf{u}}$ are compact, ψ is continuous on $\mathcal{S}_{\mathbf{w}} \times \mathcal{S}_{\mathbf{u}}$, and $\mathbf{G}, \mathbf{g}, \mathbf{h}$ **have i.i.d. standard normal entries**.

Then for any $c \in \mathbb{R}$ we have

$$\mathbb{P}(\Gamma(\mathbf{G}) < c) \le 2\mathbb{P}(\Upsilon(\mathbf{g},\mathbf{h}) \le c), \qquad \text{or} \qquad \mathbb{P}(\Gamma(\mathbf{G}) \ge c) \ge 2\mathbb{P}(\Upsilon(\mathbf{g},\mathbf{h}) > c) - 1.$$

We want to use this lemma to simplify the min-max conditions in (11) and (12)

Near Gaussianity of $\mathbf{A_z}$

- The issue is that $\mathbf{A_z}$ is non-Gaussian (otherwise [JF21] becomes trivial...)
- After a transformation, the non-Gaussianity only occurs in the first column

Lemma 3: Near Gaussianity of $\mathbf{A_z}$

Let $\mathbf{P_x}$ be an orthogonal matrix whose first row is $\mathbf{x}^\top$. Let

$$L = \frac{1}{m}\sum_{i=1}^{m} L_i, \qquad \text{where} \quad \{L_i\}_{i=1}^{m} \overset{iid}{\sim} |\mathcal{N}(0,1) + \mathcal{N}(0,1)\mathbf{i}|,$$

and let $\mathbf{G} \sim \mathcal{N}^{(m+1)\times(n-1)}(0,1)$ be independent of L. Then we have

$$\mathbf{A_z}\mathbf{P_x}^\top \overset{d}{=} \begin{bmatrix} L\mathbf{e}_1 & \frac{\mathbf{G}}{\sqrt{m}} \end{bmatrix} \tag{13}$$

$$= \begin{bmatrix} L & \frac{g_{11}}{\sqrt{m}} & \cdots & \frac{g_{1,n-1}}{\sqrt{m}} \\ 0 & \frac{g_{21}}{\sqrt{m}} & \cdots & \frac{g_{2,n-1}}{\sqrt{m}} \\ \vdots & \vdots & \ddots & \vdots \\ \underbrace{0}_{\text{nongaussian}} & \frac{g_{m+1,1}}{\sqrt{m}} & \cdots & \frac{g_{m+1,n-1}}{\sqrt{m}} \end{bmatrix} \tag{14}$$

Proof Sketches

Use Gaussian min-max to simplify:

$$\mathbb{P}(\mathbf{x}^\sharp = \mathbf{x}) \geq \mathbb{P}\Big(\min_{\mathbf{u}\in T_f^*(\mathbf{x})} \max_{\mathbf{v}\in\mathbb{S}^m} \mathbf{v}^\top \mathbf{A_z u} > 0\Big)$$ ▶ by success condition

$$= \mathbb{P}\Big(\min_{\mathbf{u}\in \mathbf{P_x} T_f^*(\mathbf{x})} \max_{\mathbf{v}\in\mathbb{S}^m} \mathbf{v}^\top \mathbf{A_z P_x^\top u} > 0\Big)$$

$$= \mathbb{P}\Big(\min_{\mathbf{u}\in \mathbf{P_x} T_f^*(\mathbf{x})} \max_{\mathbf{v}\in\mathbb{S}^m} \mathbf{v}^\top \begin{bmatrix} L\mathbf{e}_1 & \frac{\mathbf{G}}{\sqrt{m}} \end{bmatrix} \mathbf{u} > 0\Big)$$ ▶ by near-Gaussianity

$$= \mathbb{P}\Big(\min_{\mathbf{u}\in \mathbf{P_x} T_f^*(\mathbf{x})} \max_{\mathbf{v}\in\mathbb{S}^m} \sqrt{m} L u_1 v_1 + \mathbf{v}^\top \mathbf{G}\tilde{\mathbf{u}} \geq 0\Big)$$ ▶ by $\mathbf{u} = (u_1, \tilde{\mathbf{u}}^\top)^\top, \mathbf{v} = (v_1, \tilde{\mathbf{v}}^\top)^\top$

$$\geq 2\mathbb{P}\Big(\min_{\mathbf{u}\in \mathbf{P_x} T_f^*(\mathbf{x})} \max_{\mathbf{v}\in\mathbb{S}^m} \|\tilde{\mathbf{u}}\|_2 \mathbf{g}^\top \mathbf{v} + \|\mathbf{v}\|_2 \mathbf{h}^\top \tilde{\mathbf{u}} + \sqrt{m} L u_1 v_1 > \delta\Big) - 1$$ ▶ by Gaussian min-max

$$= 2\mathbb{P}\Big(\min_{\mathbf{u}\in \mathbf{P_x} T_f^*(\mathbf{x})} \mathbf{h}^\top \tilde{\mathbf{u}} + \Big\| \|\tilde{\mathbf{u}}\|_2 \mathbf{g} + \sqrt{m} L u_1 \mathbf{e}_1 \Big\|_2 > 0\Big) - 1$$

$$= 2\mathbb{P}\Big(\forall \mathbf{u} \in \mathbf{P_x} T_f^*(\mathbf{x}),\ \mathbf{h}^\top \tilde{\mathbf{u}} < \Big\| \|\tilde{\mathbf{u}}\|_2 \mathbf{g} + \sqrt{m} L u_1 \mathbf{e}_1 \Big\|_2\Big) - 1$$ ▶ by symmetry

Proof Sketches

Two Concentrations

$$\mathbb{P}\left(\mathbf{x}^\sharp = \mathbf{x}\right) \geq 2\mathbb{P}\left(\forall \mathbf{u} \in \mathbf{P}_\mathbf{x} T_f^*(\mathbf{x}),\ \mathbf{h}^\top \tilde{\mathbf{u}} < \left\| \|\tilde{\mathbf{u}}\|_2 \mathbf{g} + \sqrt{m} L u_1 \mathbf{e}_1 \right\|_2\right) - 1$$

$$\geq 2\mathbb{P}\left(\forall \mathbf{u} \in \mathbf{P}_\mathbf{x} T_f^*(\mathbf{x}),\ \frac{\mathbf{h}^\top \tilde{\mathbf{u}}}{\left(\|\tilde{\mathbf{u}}\|_2^2 + \frac{\pi u_1^2}{2}\right)^{1/2}} \leq \sqrt{m}(1-\epsilon)\right) - 1$$

- by $\left\| \|\tilde{\mathbf{u}}\|_2 \mathbf{g} + \sqrt{m} L u_1 \mathbf{e}_1 \right\|_2 \sim \sqrt{m}\left(\|\tilde{\mathbf{u}}\|_2^2 + \frac{\pi u_1^2}{2}\right)^{1/2}$

$$= 2\mathbb{P}\left(\sup_{\mathbf{u} \in \mathbf{P}_\mathbf{x} T_f^*(\mathbf{x})} \frac{\mathbf{h}^\top \tilde{\mathbf{u}}}{\left(\|\tilde{\mathbf{u}}\|_2^2 + \frac{\pi u_1^2}{2}\right)^{1/2}} \leq \sqrt{m}(1-\epsilon)\right) - 1$$

$$\approx 2\mathbb{1}\left(\sqrt{m} \geq \frac{1}{1-\epsilon} \mathbb{E} \sup_{\mathbf{u} \in \mathbf{P}_\mathbf{x} T_f^*(\mathbf{x})} \frac{\mathbf{h}^\top \tilde{\mathbf{u}}}{\left(\|\tilde{\mathbf{u}}\|_2^2 + \frac{\pi u_1^2}{2}\right)^{1/2}}\right) - 1$$

- by Gaussian concentration

Simplification:

$$\mathbb{E} \sup_{\mathbf{u} \in \mathbf{P}_\mathbf{x} T_f^*(\mathbf{x})} \frac{\mathbf{h}^\top \tilde{\mathbf{u}}}{\left(\|\tilde{\mathbf{u}}\|_2^2 + \frac{\pi u_1^2}{2}\right)^{1/2}} = \mathbb{E} \sup_{\substack{\mathbf{u} \in T_f(\mathbf{x}) \\ \|\mathbf{Q}_\mathbf{x} \mathbf{u}\|_2 = 1}} \langle (\mathbf{I}_n - \mathbf{x}\mathbf{x}^\top)\mathbf{g}, \mathbf{u} \rangle = \sqrt{\zeta_{\mathrm{PO}}(f; \mathbf{x})}$$

Connecting to Statistical Dimension

- We wish to compute

$$\zeta_{\mathrm{PO}}(\mathbf{x}; f) = \left[\mathbb{E} \sup_{\substack{\mathbf{u} \in T_f(\mathbf{x}) \\ \|\mathbf{Q}_{\mathbf{x}} \mathbf{u}\|_2 = 1}} \langle (\mathbf{I}_n - \mathbf{x}\mathbf{x}^\top)\mathbf{g}, \mathbf{u} \rangle \right]^2 \tag{15}$$

- We first approximate it by statistical dimension of the descent cone of *some signal-dependent norm*

Lemma 4: Connect to Statistical Dimension

If we define the signal-dependent norm $f_{\mathbf{x}}(\mathbf{w}) = f(\mathbf{Q}_{\mathbf{x}}^{-1}\mathbf{w})$, then we have

$$\delta\big(T_{f_{\mathbf{x}}}(\mathbf{x})\big) - \sqrt{\frac{8\delta(T_{f_{\mathbf{x}}}(\mathbf{x}))}{\pi}} - \left(1 - \frac{2}{\pi}\right) \le \zeta_{\mathrm{PO}}(\mathbf{x}; f) \le \delta\big(T_{f_{\mathbf{x}}}(\mathbf{x})\big).$$

Computing Statistical Dimension

- Then we compute $\delta\left(T_{f_{\mathbf{x}}}(\mathbf{x})\right)$ by the general recipe in [ALMT14] and have the following surrogate

$$\hat{\zeta}_{\mathrm{PO}}(\mathbf{x};f) := \inf_{\tau\geq 0} \mathbb{E}\left[\mathrm{dist}^2(\mathbf{g}, \tau\cdot\mathbf{Q}_{\mathbf{x}}^{-1}\partial f(\mathbf{x}))\right] \tag{16}$$

which can often be computed more easily

Explicit Formula – Sparse Recovery

Theorem 2: Threshold for sparse recovery

We have

$$\zeta_{\mathrm{PO}}(\mathbf{x}; \|\cdot\|_1) \sim n \cdot \psi\Big(\frac{s}{n}, \frac{\|\mathbf{x}\|_1^2}{s}\Big), \tag{17}$$

where for any $(u, v) \in (0,1) \times (0,1]$,

$$\psi(u, v) := \inf_{\tau \geq 0} \Big\{ u\Big(1 + \tau^2 - \tau^2 v(1 - \frac{2}{\pi})\Big) + (1-u)\sqrt{\frac{2}{\pi}} \int_\tau^\infty (w - \tau)^2 e^{-\frac{w^2}{2}} \, \mathrm{d}w \Big\}. \tag{18}$$

- For recovering s-sparse $\mathbf{x}$ from $\mathbf{y} = \mathbf{A}\mathbf{x}$ with $\mathbf{A} \sim \mathcal{N}^{m \times n}(0,1)$, we have

$$\zeta_{\mathrm{LN}}(\mathbf{x}; \|\cdot\|_1) \sim n \cdot \psi_1\Big(\frac{s}{n}\Big), \quad \text{where } \psi_1(u) := \inf_{\tau \geq 0} \Big\{ u(1 + \tau^2) + (1-u)\sqrt{\frac{2}{\pi}} \int_\tau^\infty (w - \tau)^2 e^{-\frac{w^2}{2}} \, \mathrm{d}w \Big\}.$$

Interesting Findings

- We have $\zeta_{\mathrm{PO}}(\mathbf{x};\|\cdot\|_1) \le \zeta_{\mathrm{LN}}(\mathbf{x};\|\cdot\|_1)$ due to $\psi(u,v) \le \psi_1(u)$
- Unlike ζ_{LN} that only depends on the sparsity s, ζ_{PO} also depends on the $\|\mathbf{x}\|_1$.
- Larger $\|\mathbf{x}\|_1$ corresponds to the smaller ζ_{PO}
  - Equal amplitude s-sparse signal

$$\mathbf{x} = \left(s^{-1/2}, s^{-1/2}, \cdots, s^{-1/2}, 0 \cdots, 0\right)^{\top}$$

  renders the earliest phase transition

Simulating $\frac{\zeta_{\text{PO}}}{\zeta_{\text{LN}}}$

- We have

$$\frac{\zeta_{\text{PO}}(\mathbf{x}; \|\cdot\|_1)}{\zeta_{\text{LN}}(\mathbf{x}; \|\cdot\|_1)} \sim \frac{\psi(\frac{s}{n}, \frac{\|\mathbf{x}\|_1^2}{s})}{\psi_1(\frac{s}{n})}, \tag{19}$$

so let us simulate

$$R_{sp}(u, v) = \frac{\psi(u, v)}{\psi_1(u)}, \quad (u, v) \in (0, 1]^2 \tag{20}$$

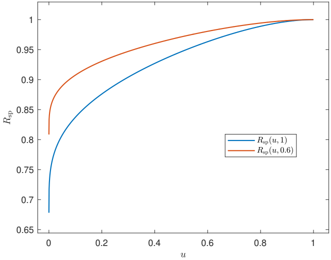


Figure 2: The left figure plots $R_{\text{sp}}(u, 1)$ and $R_{\text{sp}}(u, 0.6)$, showing $\lim_{u\to 0^+} R_{\text{sp}}(u, 1) \approx \mathbf{0.678}$ and $\lim_{u\to 0^+} R_{\text{sp}}(u, 0.6) \approx \mathbf{0.808}$.

Phase Transition Curves

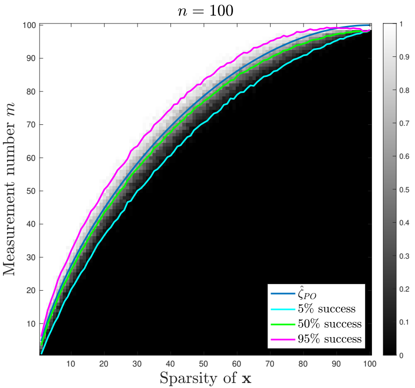


Figure 3: The empirical phase transitions of recovering equal amplitude sparse vectors in $\mathbb{R}^{100}$ are consistent with $\hat{\zeta}_{\text{PO}}(\mathbf{x}; \|\cdot\|_1)$

Dependence on ℓ_1 Norm

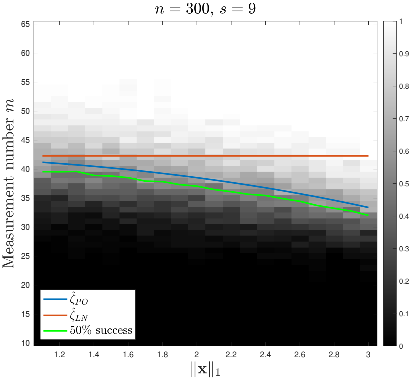


Figure 4: The empirical success rates of recovering 9-sparse signals in $\mathbb{R}^{300}$ with ℓ_1-norm varying between $[1.1,3]$, confirming earlier phase transitions under larger $\|\mathbf{x}\|_1$.

Concluding Remarks

Our work:

- We establish phase transitions for the linearization approach in PO-CS
- We compute the phase transition locations for sparse/low-rank recovery and make interesting observations (e.g., dependence on ℓ_1 norm)
- Numerical simulations back up our theory

Future direction:

- Our current theory only holds for complex Gaussian $\mathbf{\Phi}$, but we numerically observed universality over other designs. How to prove this?
- We focus on noiseless case. Can we perform precise error analysis for the noisy case?

Thank You

Thank You!